SYMMETRY-INFORMED QUANTUM METROLOGIES

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QUMINOS, Les Diablerets
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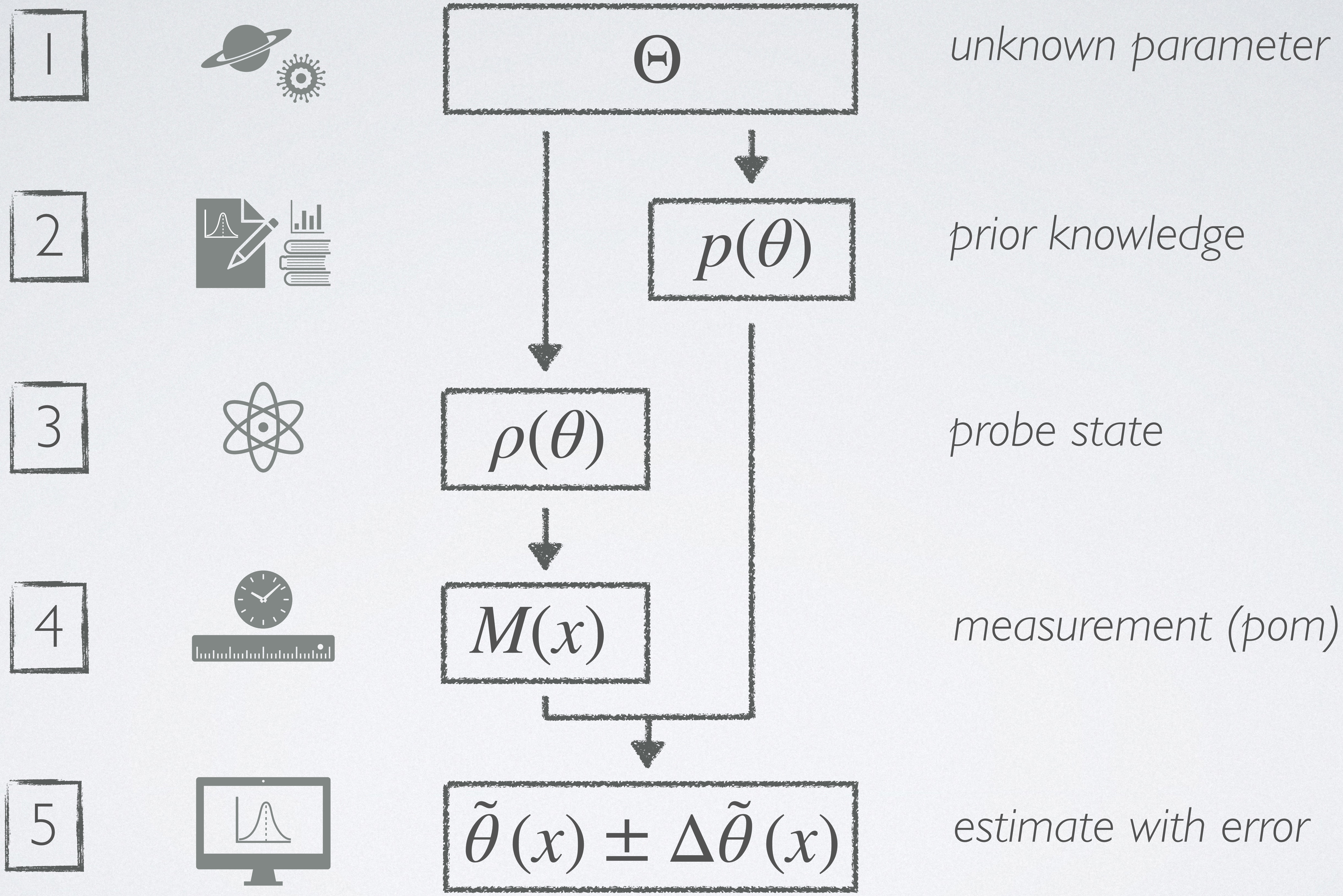
OUR PLAN FOR TODAY

- Elements of quantum metrology
- Beyond phase estimation, but why?
- Symmetry-informed metrologies
- Hyperbolic errors and the metrology of weights
- The global estimation of entanglement

ELEMENTS OF QUANTUM METROLOGY



1 Θ *unknown parameter*

2 $p(\theta)$ *prior knowledge*

3 $\rho(\theta)$ *probe state*

4 $M(x)$ *measurement (pom)*

5 $\tilde{\theta}(x) \pm \Delta\tilde{\theta}(x)$ *estimate with error*

ELEMENTS OF QUANTUM METROLOGY

Optimal quantum strategy?

(i.e., best estimator and pom?)

1 *Define*

$$W[\underbrace{\tilde{\theta}(x)}_{\text{estimator}}] = \int d\overset{\text{hypothesis}}{\theta}\, \underbrace{p(\theta)}_{\text{prior}}\, \underbrace{\rho(\theta)}_{\text{state}}\, \underbrace{\mathscr{D}[\tilde{\theta}(x), \theta]}_{\text{deviation}}$$

2 *Choose* $\mathscr{D}$

3 *Solve*

$$\boxed{\min_{\tilde{\theta},M} \bar{\epsilon} = \mathrm{Tr}\left\{ \int dx\, M(x) W[\tilde{\theta}(x)] \right\}}$$

C. W. Helstrom 1976, Academic Press, New York.

BEYOND PHASE ESTIMATION, BUT WHY?

	support	$\mathscr{D}$	$\bar{\epsilon}$	$\tilde{\theta}$	*pom*
local		$(\tilde{\theta}-\theta)^2$	$\frac{1}{\mu\mathscr{F}}$	MLE (typically)	$F \simeq \mathscr{F}$
↑		$(\tilde{\theta}-\theta)^2$	mean squared	posterior average	Personick's projectors
global		$4\sin^2\left(\frac{\tilde{\theta}-\theta}{2}\right)$	mean sinusoidal	covariant measurements	

J. Phys. A: Math. Theor. 53 363001 (2020)
A. S. Holevo 2011, Edizioni della Normale, Springer Basel.
C. W. Helstrom 1976, Academic Press, New York.

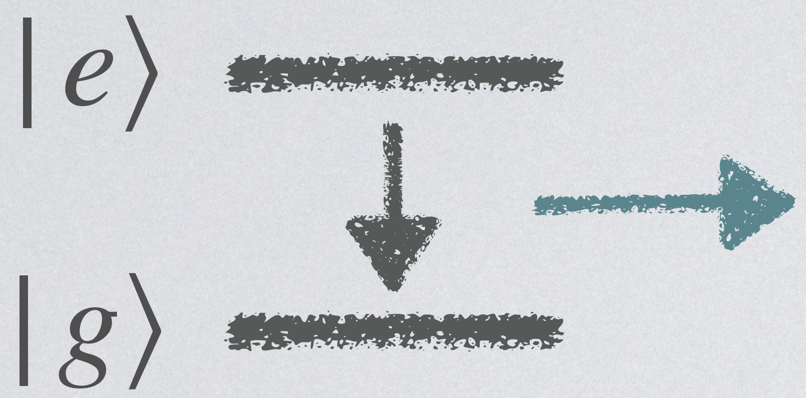
BEYOND PHASE ESTIMATION, BUT WHY?

Quantum estimation of a lifetime

Let a two-level atom

$$|\psi\rangle = \sqrt{1-a}\,|e\rangle + \sqrt{a}\,|g\rangle$$

undergo spontaneous photon emission



so that

$$\rho_t(\tau) = \begin{pmatrix} \left[1 - a\,\mathrm{e}^{-t/\tau}\right] & \left[a(1-a)\,\mathrm{e}^{-t/\tau}\right]^{\frac{1}{2}} \\ \left[a(1-a)\,\mathrm{e}^{-t/\tau}\right]^{\frac{1}{2}} & a\,\mathrm{e}^{-t/\tau} \end{pmatrix}.$$

What is the value of the the lifetime τ?

Scale invariance

Let be θ a hypothesis about τ; if we are <u>maximally ignorant</u>, our knowledge is invariant under

$$\theta \to \theta' = \gamma\theta$$
$$t \to t' = \gamma t$$

i.e., $t'/\theta' = t/\theta$.

Logarithmic error

$$\boxed{\mathscr{D}(\tilde{\theta}, \theta) = \log^2(\tilde{\theta}/\theta)}$$

- *Symmetric:* $\mathscr{D}(\tilde{\theta}, \theta) = \mathscr{D}(\theta, \tilde{\theta})$
- *Invariant:* $\mathscr{D}(\gamma\tilde{\theta}, \gamma\theta) = \mathscr{D}(\tilde{\theta}, \theta)$
- *Monotonic growth (decrease) as* $\tilde{\theta} > \theta$ $(\tilde{\theta} < \theta)$, *vanishing at* $\tilde{\theta} = \theta$.

Quantum Sci. Technol. 8, 015009 (2022); PRL 127, 190402 (2021)

BEYOND PHASE ESTIMATION, BUT WHY?

Quantum scale estimation

Let $\mathcal{S} = \int ds\, \mathscr{P}(s)\, s$ *be solution to* $\mathcal{S}\rho_0 + \rho_0\mathcal{S} = 2\rho_1$,

where $\rho_k = \int d\theta p(\theta)\rho(\theta)\log^k(\theta);$ *then, the optimal estimator and pom are*

$$\boxed{\tilde{\theta}(s) = \exp(s),\ M(s) = \mathscr{P}(s)}$$

and the minimum mean logarithmic error is

$$\boxed{\bar{\epsilon}_{\min} = \int d\theta p(\theta)\log^2(\theta) - \mathrm{Tr}(\rho_0\mathcal{S}^2)}$$

prior info. *info. gain*

Quantum Sci. Technol. 8, 015009 (2022)

PRL 127, 190402 (2021)

SYMMETRY-INFORMED METROLOGIES

Scale estimation	Symmetry-informed estimation
Invariant under transformations	
$\theta \to \theta' = \gamma\theta$	$\theta \to \theta' = g(\theta)$
Mapping to location estimation	
$\log(\theta) \to \log(\theta') = \log(\theta) + \log(\gamma)$	$f(\theta) \to f(\theta') = f(\theta)+c$
$\mathscr{D}(\tilde{\theta}, \theta) = \left[\log\left(\frac{\tilde{\theta}}{\theta_u}\right) - \log\left(\frac{\theta}{\theta_u}\right)\right]^2$	$\mathscr{D}(\tilde{\theta}, \theta) = [f(\tilde{\theta}) - f(\theta)]^2$ quadratic errors

PRL 127, 190402 (2021)
Phys. Rev. A 105, 042601 (2022)

SYMMETRY-INFORMED METROLOGIES

Optimal quantum strategy for quadratic errors

Let $\mathcal{S} = \int ds\, \mathcal{P}(s)\, s$ *be solution to* $\mathcal{S}\rho_0 + \rho_0\mathcal{S} = 2\rho_1$,

where $\rho_k = \int d\theta p(\theta)\rho(\theta) f^k(\theta)$; *then, the* *<u>optimal estimator</u>*

<u>and pom</u> are

$$\boxed{\tilde{\theta}(s) = f^{-1}(s),\ \ M(s) = \mathcal{P}(s)}$$

and the <u>minimum mean quadratic error</u> is

$$\boxed{\bar{\epsilon}_{\min} = \underbrace{\int d\theta p(\theta) f^2(\theta)}_{\text{prior info.}} - \underbrace{\mathrm{Tr}(\rho_0 \mathcal{S}^2)}_{\text{info. gain}}}$$

arXiv:2402.16410

Quantum Sci. Technol. 8, 015009 (2022)

HYPERBOLIC ERRORS AND THE METROLOGY OF WEIGHTS

Probability of success

$$p(r\,|\,n,\eta) = B(n,r)\,\eta^r(1-\eta)^{n-r}$$

Mixture of two quantum states

$$\varrho = \eta\,\varrho_0 + (1-\eta)\,\varrho_1$$

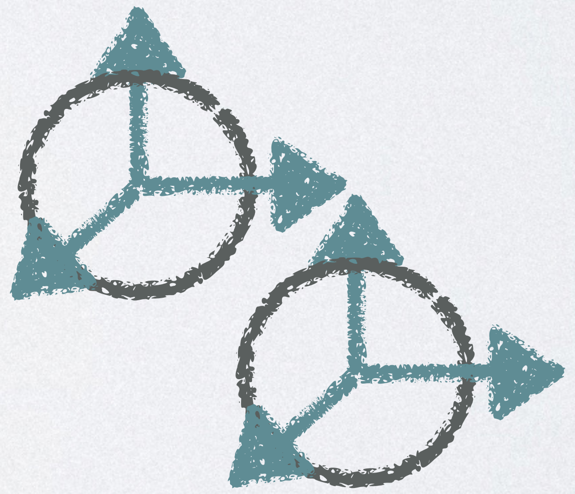
Weight parameters

$$\eta \in (0,1)$$

Measure the relative weight between two possibilities as η and $1-\eta$.

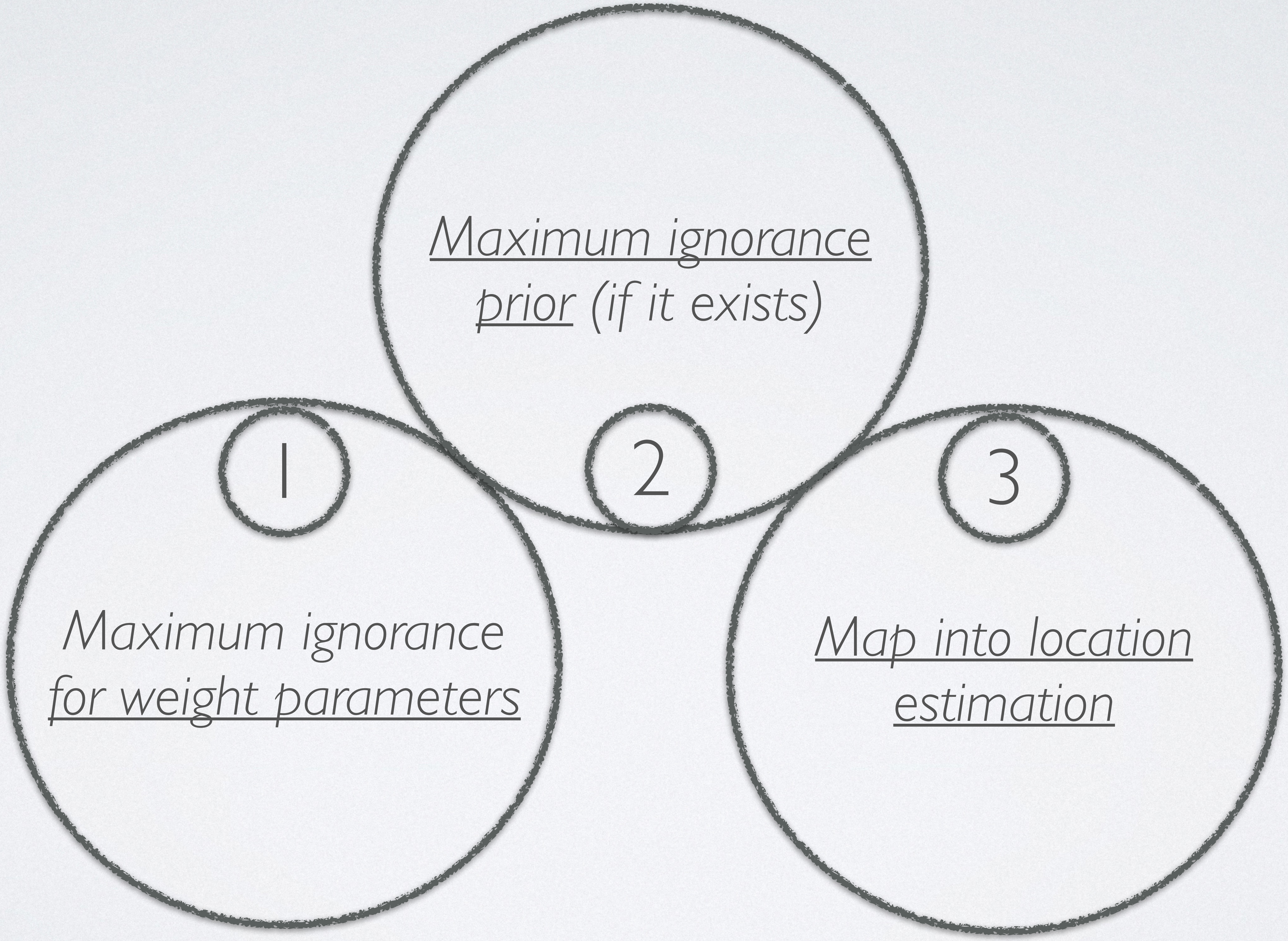
Schmidt parameter characterising the two-qubit pure state family

$$|\psi\rangle = \sqrt{\eta}\,|e_0\rangle_A \otimes |e_0\rangle_B + \sqrt{1-\eta}\,|e_1\rangle_A \otimes |e_1\rangle_B$$



HYPERBOLIC ERRORS AND THE METROLOGY OF WEIGHTS

How do we construct a quantum metrology for weight parameters <u>using symmetry-informed estimation</u>?



<u>Maximum ignorance prior</u> (if it exists)

Maximum ignorance <u>for weight parameters</u>

<u>Map into location estimation</u>

PRL 127, 190402 (2021)

HYPERBOLIC ERRORS AND THE METROLOGY OF WEIGHTS

Weight parameters

Measure the relative weight between two possibilities, e_0 *and* e_1*, as* η *and* $1-\eta$.

- *My hypothesis is* θ.
- *The probability I would choose* e_0 *is* $p(e_0) = \theta$.

- *My hypothesis is* θ'.
- *The probability I would choose* e_0 *is* $p(e_0 \mid I) = \theta'$.

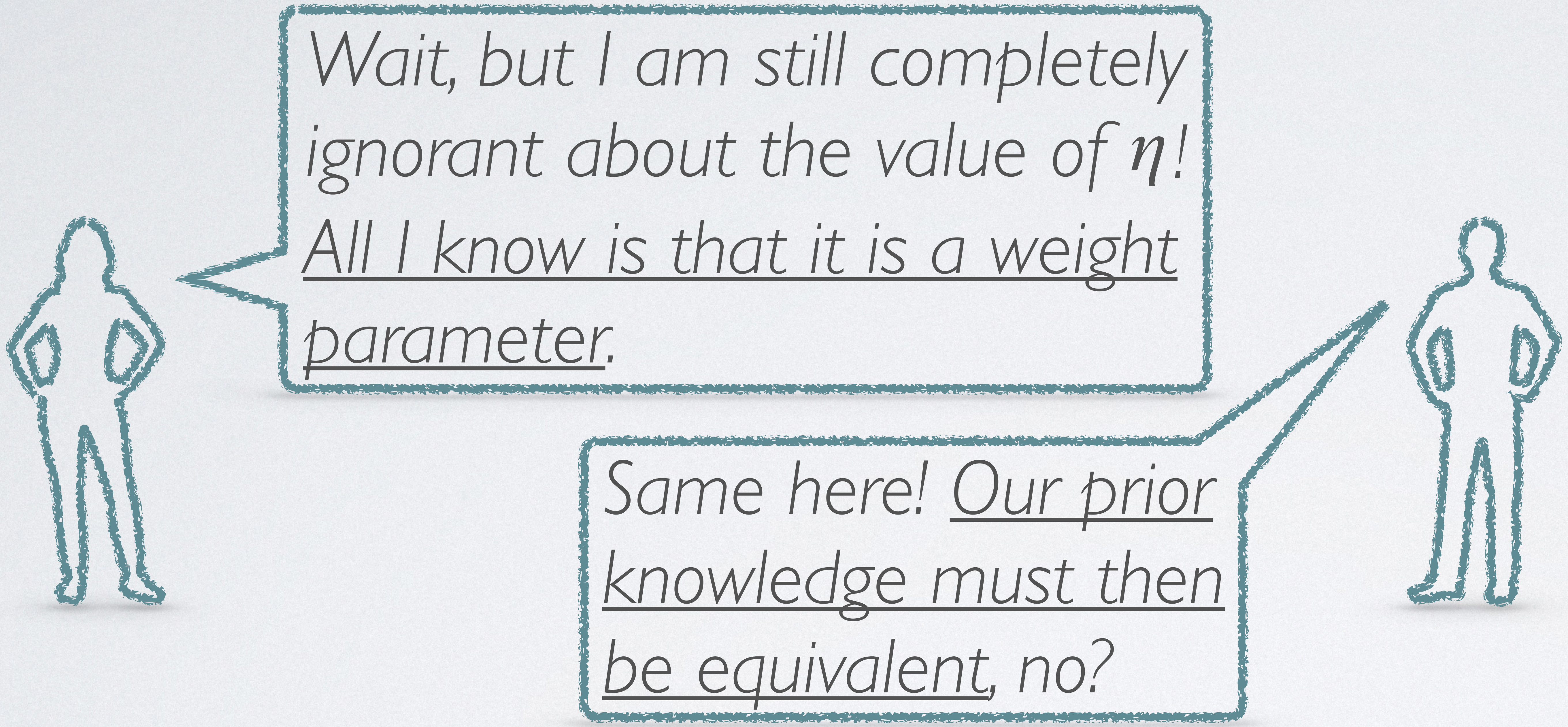
$$p(e_0 \mid I) = \frac{p(e_0)p(I \mid e_0)}{p(e_0)p(I \mid e_0) + p(e_1)p(I \mid e_1)} \longrightarrow$$

Möbius transformation

$$\theta' = \frac{\gamma\theta}{1 - \theta + \gamma\theta}$$

IEEE Trans Cybern 4, 227-241 (1968)

HYPERBOLIC ERRORS AND THE METROLOGY OF WEIGHTS



Wait, but I am still completely ignorant about the value of η! All I know is that it is a weight parameter.

Same here! Our prior knowledge must then be equivalent, no?

$$p(\theta')d\theta' = p(\theta)d\theta$$

(functional equation)

↓

Haldane's prior

$$p(\theta) \propto \frac{1}{\theta(1-\theta)}$$

IEEE Trans Cybern 4, 227-241 (1968)

HYPERBOLIC ERRORS AND THE METROLOGY OF WEIGHTS

Let φ *be a location parameter and* $\mathscr{D}(\tilde{\varphi}, \varphi) = (\tilde{\varphi} - \varphi)^2$.
If our problem is *isomorphic to location estimation, then,*
in the limit of maximum ignorance,

$$p(\varphi) d\varphi = p(\theta) d\theta \quad \longrightarrow \quad d\varphi \propto \frac{d\theta}{\theta(1-\theta)}$$

Therefore:

Hyperbolic error	*Symmetry function*
$D(\tilde{\theta}, \theta) = 4 \operatorname{artanh}^2 \left(\frac{\tilde{\theta} - \theta}{\tilde{\theta} + \theta - 2\tilde{\theta}\theta} \right)$	$f(z) = 2 \operatorname{artanh}(2z - 1)$

arXiv:2402.16410

HYPERBOLIC ERRORS AND THE METROLOGY OF WEIGHTS

Quantum weight estimation

Let $\mathcal{S} = \int ds\, \mathscr{P}(s)\, s$ *be solution to* $\mathcal{S}\rho_0 + \rho_0\mathcal{S} = 2\rho_1$,

where $\rho_k = 2^k \int d\theta p(\theta)\rho(\theta)\,\mathrm{artanh}^k(2\theta - 1)$; *then, the optimal estimator and pom are*

$$\boxed{\tilde{\theta}(s) = [1 + \tanh(s/2)]/2,\ M(s) = \mathscr{P}(s)}$$

logistic function

and the minimum mean quadratic error is

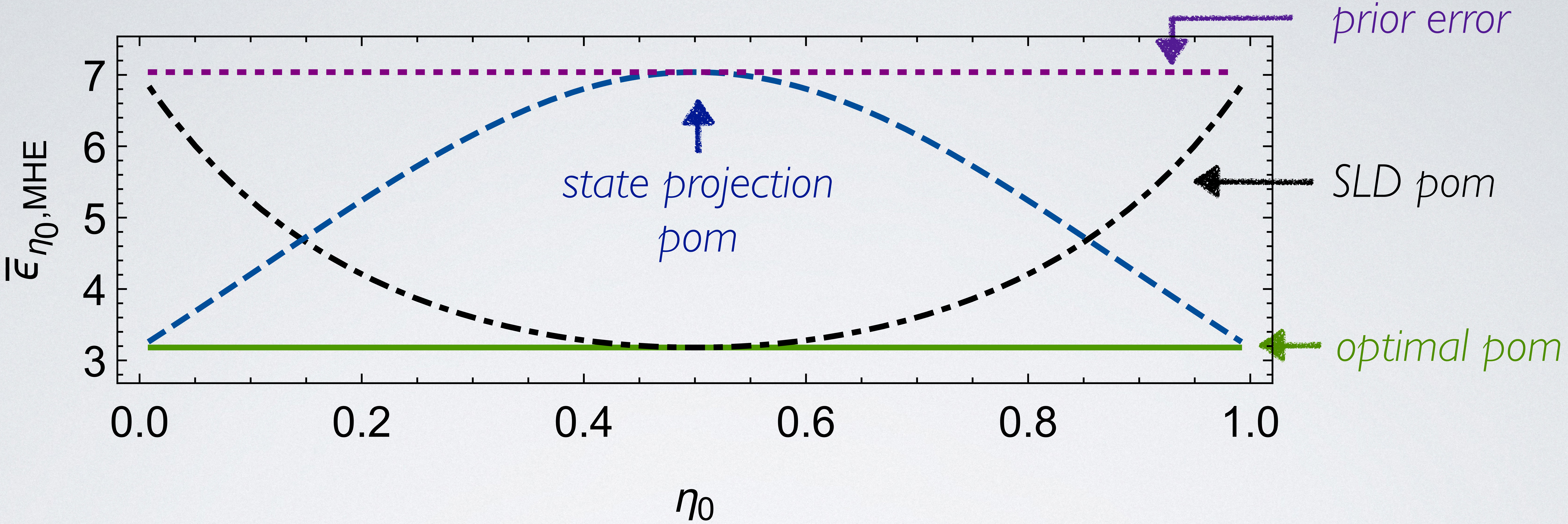
$$\boxed{\bar{\epsilon}_{\min} = 4\int d\theta p(\theta)\,\mathrm{artanh}^2(2\theta - 1) - \mathrm{Tr}(\rho_0\mathcal{S}^2)}$$

prior info.

info. gain

arXiv:2402.16410

THE GLOBAL ESTIMATION OF ENTANGLEMENT



Optimal pom	*SLD pom*	*State projection pom*
Eigenbasis of $\mathcal{S} \propto \sigma_3$	*Eigenbasis of* $\mathcal{L} = 2\partial_z \rho(z)\,\vert_{\eta_0}$	$\vert\psi(\eta_0)\rangle, \vert\psi_\perp(\eta_0)\rangle$

$F(\theta) = \mathscr{F}(\theta)$

TAKE-HOME MESSAGE

Parameter	phase	location	scale	weight	location-isomorphic
Support	$0 < \theta < 2\pi$	$-\infty < \theta < \infty$	$0 < \theta < \infty$	$0 < \theta < 1$	$-\infty < f(\theta) < \infty$
Symetry	$\theta' = \theta + 2n\pi$	$\theta' = \theta + c$	$\theta' = \gamma\theta$	$\theta' = \dfrac{\gamma\theta}{1 - \theta + \gamma\theta}$	$f(\theta') = f(\theta) + c$
Ignorance prior	$p(\theta) = \frac{1}{2\pi}$	$p(\theta) \propto 1$	$p(\theta) \propto \frac{1}{\theta}$	$p(\theta) \propto \frac{1}{\theta(1-\theta)}$	$p(\theta) \propto \frac{df(\theta)}{d\theta}$
Error	$4\sin^2\left(\frac{\tilde{\theta} - \theta}{2}\right)$	$(\tilde{\theta} - \theta)^2$	$\log^2\left(\frac{\tilde{\theta}}{\theta}\right)$	$4\,\mathrm{artanh}^2\left(\frac{\tilde{\theta} - \theta}{\tilde{\theta} + \theta - 2\tilde{\theta}\theta}\right)$	$[f(\tilde{\theta}) - f(\theta)]^2$

To learn more: arXiv:2402.16410