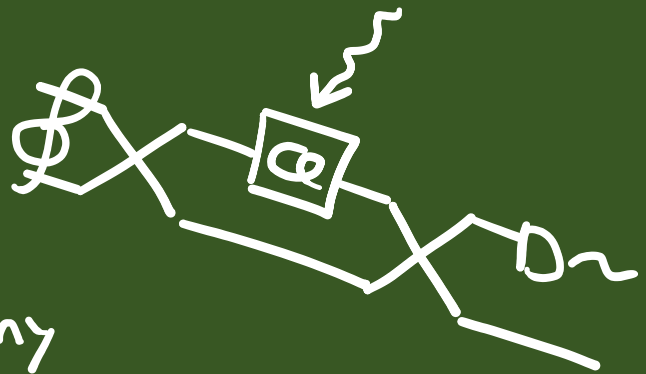
Variational principles in quantum sensing

$$d_\alpha \Sigma(A+\alpha\Omega)|_{\alpha=0} = 0$$



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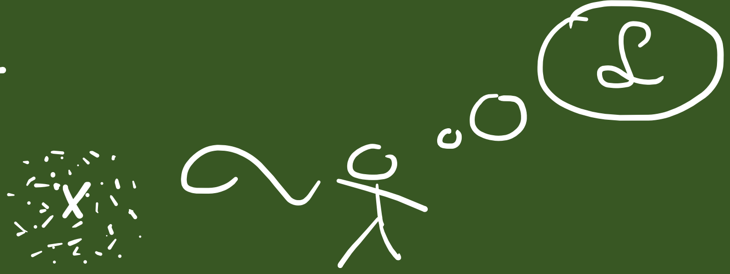
Quantum retreat
(Somerset)

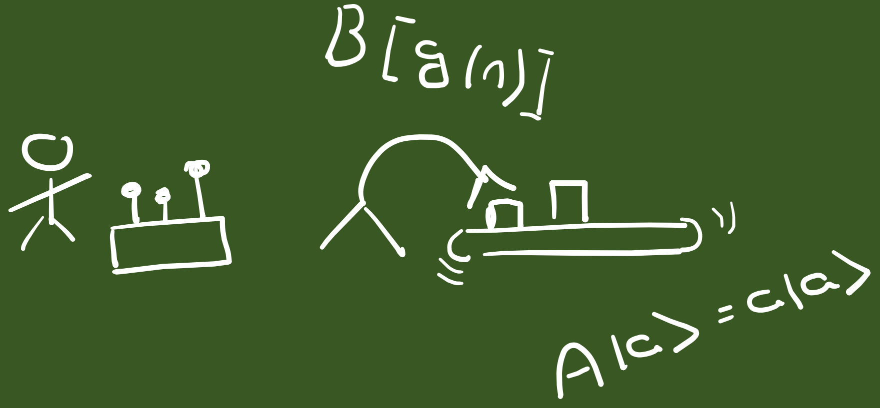
Our plan for today:

- To measure is to know

- Lagrangians, actions, and all that
- Taming uncertainty



- Working with functionals
- Did you say operators?

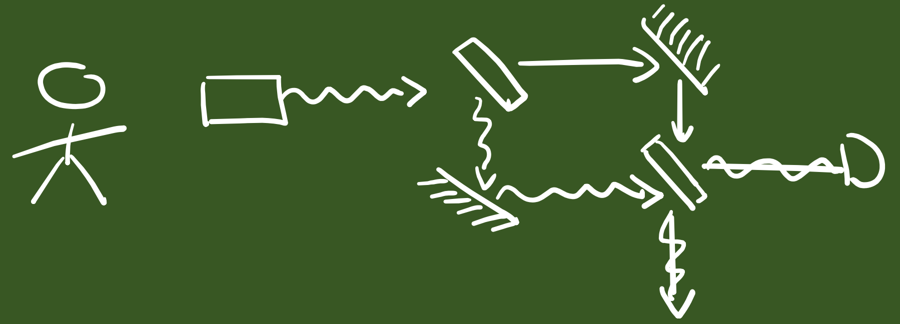


* Practice, practice, practice

To measure is to know (and knowledge is everything)

- What is a measurement?

  ↳ Collection of operations/actions on physical systems such that a set of numbers is rendered



- Numbers $\xrightarrow[\text{quantify}]{}$ properties $\xrightarrow[\text{which can be related to find}]{}$ laws of nature

- Measuring $\equiv$ interrogating nature

- Types of measurement:

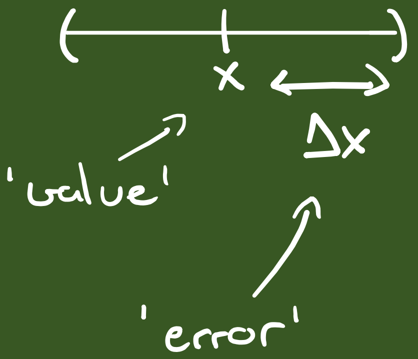
  → <u>direct</u>

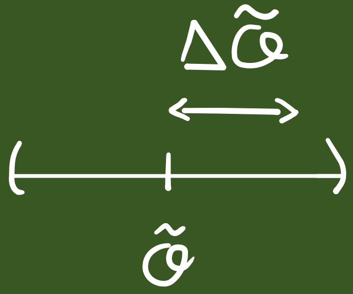
   x

  → <u>indirect</u>:

   $x \longmapsto \tilde{\Theta}(x) \equiv$ <u>estimate</u> for some quantity Θ

- <u>Ambiguity, aka uncertainty</u>:





How do we keep the uncertainty to a minimum?

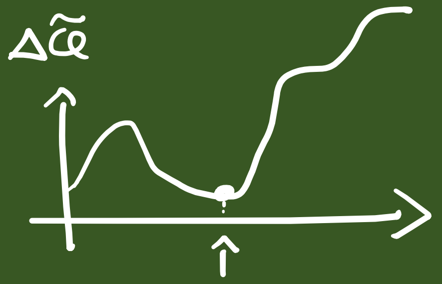
Taming uncertainty

- Bounds : $\Delta\tilde{\mathcal{O}} \geq \ldots$

  Pros : typically analytical

  Cons: tend to rely on strong assumptions (e.g., CRB)

- Numerics : brute force, probabilistic, adaptive ...



  Pros: accessible and universally applicable

  Cons: beyond visuals, tend to lack explanatory power

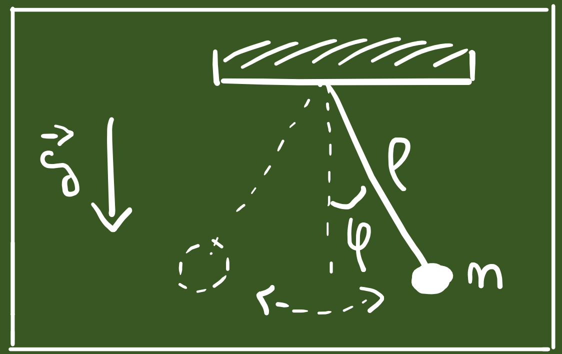
- Calculus of variations:

  Pros: fundamental, systematic, general, physical

  Cons: sometimes difficult to solve

think first; compute later

INTERLUDE: Lagrangians, actions, and all that

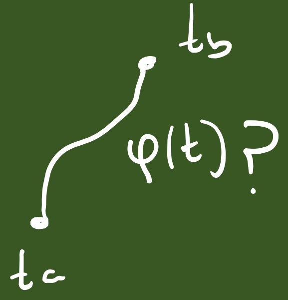


- **Degree of freedom:** φ

- We seek: $\varphi(t) \equiv$ dynamics (m, ℓ, g known)

- **Action:**

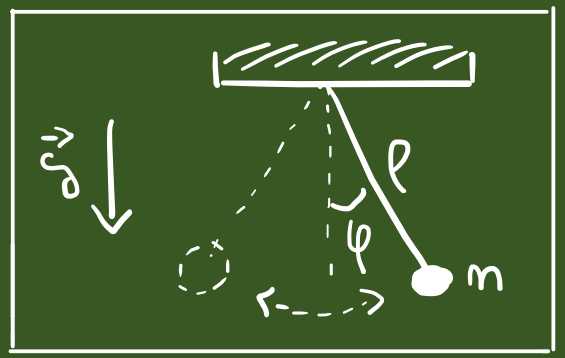
$$S = \int_{t_a}^{t_b} dt\, \mathcal{L}(t, \varphi, \dot{\varphi})$$



- **Lagrangian:**

$$\mathcal{L}(t, \varphi, \dot{\varphi}) = T - V = \frac{1}{2} m \ell^2 \dot{\varphi}^2 + mg\ell \cos\varphi$$

INTERLUDE: Lagrangians, actions, and all that



- Hamilton's principle of stationary action:

$$\varphi(t) \text{ s.t. } \delta S = 0$$

- Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \quad \Rightarrow \quad \ddot{\varphi} + \frac{g}{\ell}\sin\varphi = 0$$

- Approximate solution:

If $\varphi \ll 1$, $\sin\varphi \simeq \varphi$

$$\Rightarrow \quad \ddot{\varphi} + \frac{g}{\ell}\varphi = 0$$

$$\Rightarrow \quad \boxed{\varphi(t) = A\sin\left(\sqrt{\frac{g}{\ell}}\,t\right) + B\cos\left(\sqrt{\frac{g}{\ell}}\,t\right)}$$

Taming uncertainty

- Time t $\longleftrightarrow$ Measurand x
- Degree of freedom $\varphi(t)$ $\longleftrightarrow$ Estimator $\tilde{\Theta}(x)$
- Action S $\longleftrightarrow$ Uncertainty $\Delta\tilde{\Theta}_D = \int dx\, L[x, \tilde{\Theta}(x)]$
- Lagrangian $\mathcal{L}$ $\longleftrightarrow$ $L = p(x) \int d\Theta\, p(\Theta|x)\, D[\tilde{\Theta}(x), \Theta]$

evidence

posterior probability (Bayes theorem)

deviation function

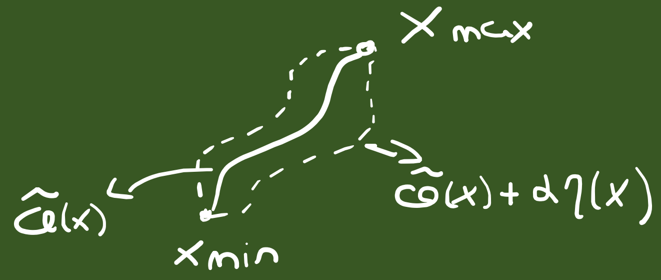
hypothesis about unknown Θ

$\boxed{\text{CALCULUS OF VARIATIONS}}$

* How do we get an analogue of the Euler-Lagrange eq.?

↳ Since $\Delta\tilde{\omega}_D$ is a functional of $\tilde{\omega}(x)$, we need to solve

$$\boxed{\frac{d}{d\alpha}\Delta\tilde{\omega}_D\left[\tilde{\omega}(x)+\alpha\,\eta(x)\right]\Big|_{\alpha=0}=0, \quad \forall\,\eta(x)}$$



for $\tilde{\omega}(x)$.

* How do we check that such estimator gives rise to a minimum?

$$\boxed{\frac{d^2}{d\alpha^2}\Delta\tilde{\omega}_D\left[\tilde{\omega}(x)+\alpha\,\eta(x)\right]\Big|_{\alpha=0}>0, \quad \forall\,\eta(x)}$$

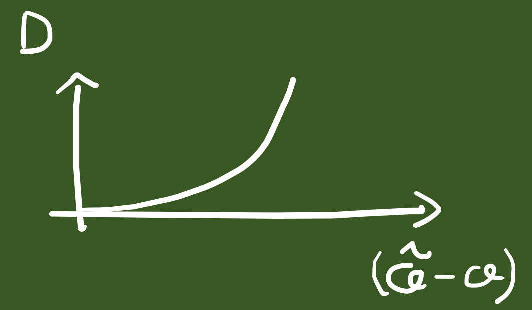
(use with care!)

Working with functionals

* An old friend: <u>the mean square error</u> (MSE)

    - Deviation function:

$$D(\hat{\alpha}, \alpha) = (\tilde{\alpha} - \alpha)^2$$



    - Analogue of the Lagrangian:

$$L = p(x) \int d\alpha \, p(\alpha | x) \left[\tilde{\alpha}(x) - \alpha \right]^2$$

    - MSE:

$$\Delta \tilde{\alpha}^2 [\tilde{\alpha}(x)] = \int dx \, p(x) \int d\alpha \, p(\alpha | x) \left[\tilde{\alpha}(x) - \alpha \right]^2$$

- Calculate:

$$\frac{d}{d\alpha} \Delta \tilde{\omega}^2 [\tilde{\omega}(x) + \alpha \eta(x)] \Big|_{\alpha=0}$$

$$= \frac{d}{d\alpha} \int dx\, p(x) \int d\omega\, p(\omega | x) \, [\tilde{\omega}(x) + \alpha \eta(x) - \omega]^2 \Big|_{\alpha=0}$$

$$= 2 \int dx\, p(x) \int d\omega\, p(\omega | x) \, [\tilde{\omega}(x) + \alpha \eta(x) - \omega] \, \eta(x) \Big|_{\alpha=0}$$

$$= 2 \int dx\, p(x) \int d\omega\, p(\omega | x) \, [\tilde{\omega}(x) - \omega] \, \eta(x)$$

$$= \int dx \left\{ 2 p(x) \int d\omega\, p(\omega | x) \, [\tilde{\omega}(x) - \omega] \right\} \eta(x)$$

- Imposing

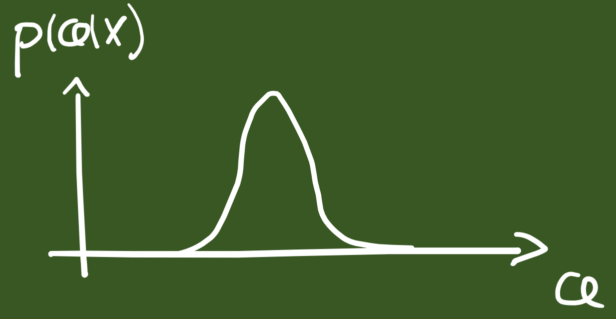
$$\frac{d}{d\alpha} \Delta \tilde{\omega}^2 [\tilde{\omega}(x) + \alpha \eta(x)] \Big|_\alpha = 0, \quad \forall \eta(x)$$

$$\Rightarrow \qquad 2p(x) \int d\omega \, p(\omega|x) [\omega - \tilde{\omega}(x)] = 0$$

$$\Rightarrow \qquad \boxed{\int d\omega \, p(\omega|x) [\omega - \tilde{\omega}(x)] = 0} \longrightarrow \text{Euler-Lagrange eq. analogue}$$

- Solving for $\tilde{\omega}(x)$:

$$\boxed{\tilde{\omega}(x) = \int d\omega \, p(\omega|x) \, \omega}$$



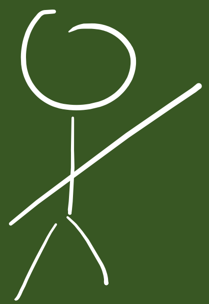
- Does it give rise to a minimum?

$$\frac{d^2}{d\alpha^2} \Delta \tilde{\theta}^2\left[\tilde{\theta}(x) + \alpha\,\eta(x)\right]\Big|_{\alpha} = 2\int dx\, p(x) \overbrace{\int d\theta\, p(\theta|x)}^{=1} \eta(x)^2$$

$$= 2\int dx\, p(x)\, \eta(x)^2 > 0$$

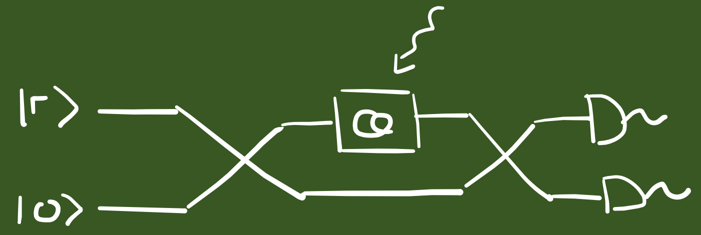
$\Rightarrow$

$$\boxed{\begin{array}{c} \tilde{\theta}(x) = \int d\theta\, p(\theta|x)\,\theta \\ \text{is the optimal estimator} \\ \text{for the square error} \\ \text{criterion} \end{array}}$$

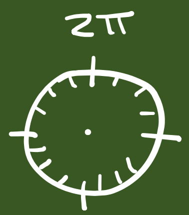


* Is it really that simple?

↳ <u>non-separable errors</u>



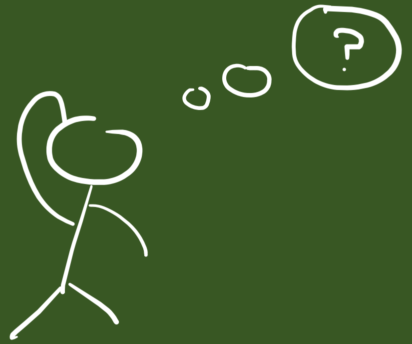
- In <u>phase estimation:</u> $D(\tilde{\theta}, \theta) = 4\sin^2\left(\frac{\tilde{\theta}-\theta}{2}\right)$

$\theta \in$ 

- This leads to the condition (same calculation as before)

$$\boxed{\int d\theta \; p(\theta|x)\, \sin\left[\tilde{\theta}(x) - \theta\right] = 0}$$

for $\tilde{\theta}(x)$.

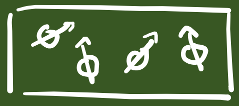


- It is not obvious how to solve this for $\tilde{\theta}(x)$!

Did you say operators?

- In quantum experiments (with ideal measurements):

outcome

 …  … x_i $\longrightarrow$ $p(x_i|\alpha) = \mathrm{Tr}\left[|x_i\rangle\langle x_i|\rho(\alpha)\right]$

$|x_i\rangle\langle x_i|$

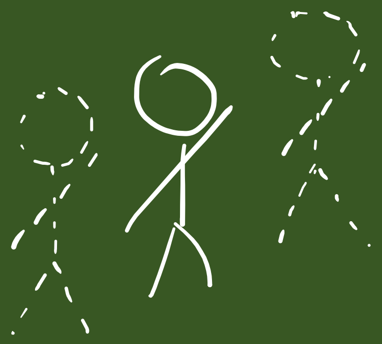
$\rho(\alpha)$

density operator

parameter

probability-operator measurement (POM)

outcome statistics given by the Born rule



- <u>Uncertainty</u> (MSE):

$$\Delta\tilde{\omega}^2 = \sum_i p(x_i) \int d\omega \, p(\omega|x_i) \left[\tilde{\omega}(x_i) - \omega\right]^2$$

$$= \sum_i \int d\omega \, p(x_i, \omega) \left[\tilde{\omega}(x_i) - \omega\right]^2$$

$$= \sum_i \int d\omega \, p(\omega) \, p(x_i|\omega) \left[\tilde{\omega}(x_i) - \omega\right]^2$$

<u>prior probability</u>

$$= \sum_i \int d\omega \, p(\omega) \, \mathrm{Tr}\left[|x_i\rangle\langle x_i| \, \rho(\omega)\right] \left[\tilde{\omega}(x_i) - \omega\right]^2$$

$$= \sum_i \int d\omega \, p(\omega) \, \mathrm{Tr}\left[|x_i\rangle\langle x_i| \, \rho(\omega)\right] \left[\omega^2 + \tilde{\omega}^2(x_i) - 2\omega\tilde{\omega}(x_i)\right] \equiv [*]$$

Define

- $\rho_k := \int d\omega \, p(\omega) \, \rho(\omega) \, \omega^k$

- $A_k := \sum_i |x_i\rangle\langle x_i| \, \tilde{\omega}(x_i)^k$

Then,

$$[*] = \mathrm{Tr}\left(\rho_2 + \rho_0 A_2 - 2\rho_1 A_1\right)$$

$$= \boxed{\mathrm{Tr}\left(\rho_2 + \rho_0 A_1^2 - 2\rho_1 A_1\right) \equiv \Delta\tilde{\omega}^2(A_1)}$$

This holds as

$$A_2 = \sum_i |x_i\rangle\langle x_i| \, \tilde{\omega}(x_i)^2$$

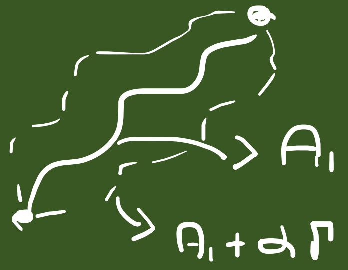
$$= \sum_{i,j} \delta_{ij} |x_i\rangle\langle x_j| \, \tilde{\omega}(x_i)\tilde{\omega}(x_j)$$

$$= \left[\sum_i |x_i\rangle\langle x_i| \, \tilde{\omega}(x_i)\right]\left[\sum_j |x_j\rangle\langle x_j| \, \tilde{\omega}(x_j)\right] = A_1^2$$

- For fixed prior $p(\Theta)$ and state $\rho(\Theta)$, we can now minimise $\Delta\tilde{\Theta}^2$ with respect to <u>both</u>

  $\rightarrow$ the estimator $\tilde{\Theta}(x_i)$, <u>and</u>

  $\rightarrow$ the POM $|x_i\rangle\langle x_i|$.



- $\tilde{\Theta}(x_i)$ and $|x_i\rangle\langle x_i|$ inside $A_1 = \sum_i |x_i\rangle\langle x_i| \tilde{\Theta}(x_i)$

- Therefore,

$$\frac{d}{d\alpha}\Delta\tilde{\Theta}^2(A_1 + \alpha\Gamma)\Big|_{\alpha=0} = \frac{d}{d\alpha}\mathrm{Tr}\Big\{\rho_2 + \rho_0\left[A_1^2 + \alpha^2\Gamma^2 + \alpha(A_1\Gamma + \Gamma A_1)\right] - 2\rho_1(A_1 + \alpha\Gamma)\Big\}\Big|_{\alpha=0}$$

$$= \mathrm{Tr}\left[2\alpha\rho_0\Gamma^2 + \rho_0(A_1\Gamma + \Gamma A_1) - 2\rho_1\Gamma\right]\Big|_{\alpha=0}$$

$$= \mathrm{Tr}\left[(A_1\rho_0 + \rho_0 A_1 - 2\rho_1)\Gamma\right] = 0$$

- By imposing

$$\frac{d}{d\alpha}\,\Delta\tilde{\hat{O}}^{2}(A_1+\alpha\Gamma)\Big|_{\alpha=0} = 0\,, \qquad \forall\,\Gamma$$

we conclude that the optimal strategy (estimator + POM) must be such that

$$A_1 = \sum_i |x_i\rangle\langle x_i|\,\tilde{\hat{O}}(x_i) = S\,,$$

where S is solution to

$$\boxed{S\rho_0 + \rho_0 S = 2\rho_1}\,.$$

Practice, practice, practice

☒ **Exercise 1:** Find the optimal estimator for the logarithmic deviation function $D(\tilde{\varphi},\varphi) = \log^2(\tilde{\varphi}/\varphi)$.

☒ **Exercise 2:** For the MSE, show that $A_1 = S$, where S is solution to $S\rho_0 + \rho_0 S = 2\rho_1$, gives rise to a minimum.

☒ **Problem:** Let the statistics of a quantum interferometer be given by

$$p(0|\varphi) = \cos^2\left(\frac{\varphi}{2}\right), \quad p(1|\varphi) = \sin^2\left(\frac{\varphi}{2}\right),$$

with $\varphi \in [0, \pi/2]$ and $p(\varphi) = 2/\pi$. Solve

$$\int_0^{\pi/2} d\varphi \, p(\varphi|i) \sin(\tilde{\varphi}_i - \varphi) = 0, \quad i = 0, 1$$

for $\tilde{\varphi}_i$.