From networks to thermometry: precision in quantum technologies

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Key works:

arXiv:2111.11921
Phys. Rev. Lett. **127**, 190402 (2021)
J. Phys. A: Math. Theor., 53 344001 (2020)
JPhys. Rev. A 101, 032114 (2020)

QUINFOG seminar
Instituto de Física Fundamental

14th Dec 2021

Our plan for today

0. Quantum estimation theory *à la* Bayes

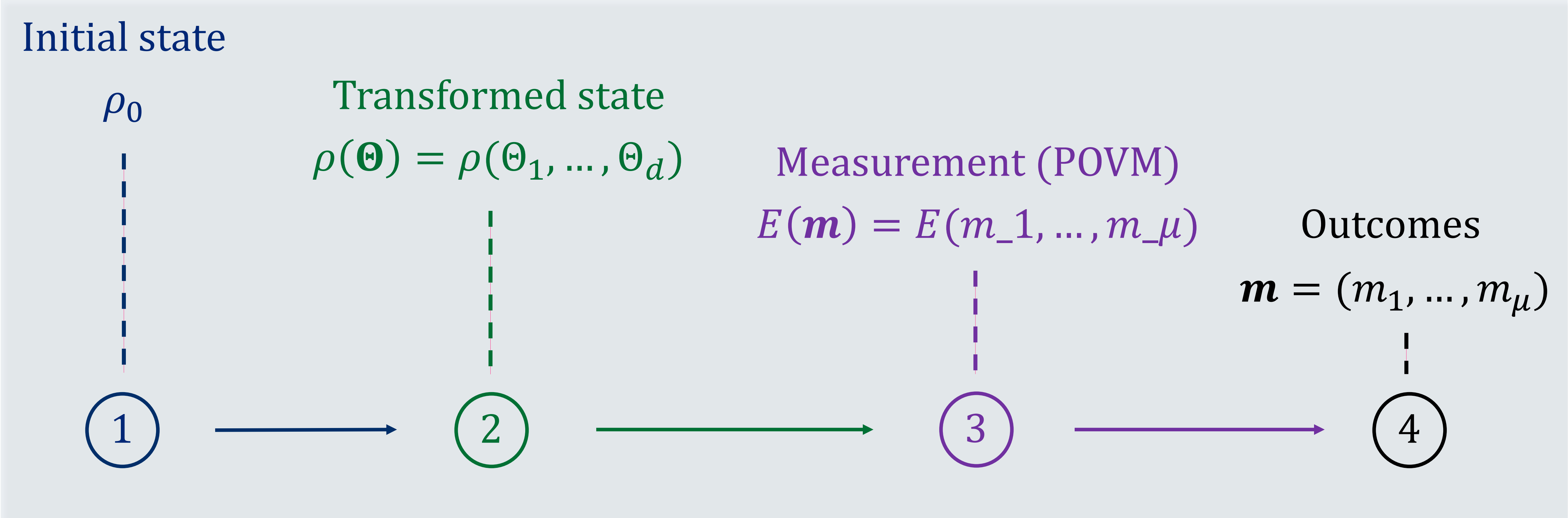
I. Quantum sensing networks: imaging and qubits
   - > Local and global properties
   - > Geometry and correlations

II. Quantum thermometry – a tale in three acts:
   - > the practical,
   - > the local,
   - > and the global

III. Quantum metrology of scale parameters
   - > Phases, locations and … scales?
   - > How does nature do it: optimal strategies in scale estimation

Quantum Estimation Theory à la Bayes

Preparation and measurement



$\boldsymbol{\theta} \equiv$ hypothesis about the true but unknown values $\boldsymbol{\Theta}$

Estimation within the Bayesian paradigm

- Prior information: $p(\boldsymbol{\theta})$
  - > Irrespective of the measurement outcomes
  - > In many cases: maximum ignorance

- Likelihood function: $p(\boldsymbol{m}|\boldsymbol{\theta}) = \mathrm{Tr}[E(\boldsymbol{m})\rho(\boldsymbol{\theta})]$
  - > Links the unknown parameter with the measured quantity
  - > Given by the Born rule (quantum systems)

- **Bayes theorem**

$$p(\boldsymbol{\theta}|\boldsymbol{m}) \propto p(\boldsymbol{\theta})\, p(\boldsymbol{m}|\boldsymbol{\theta}) = p(\boldsymbol{\theta})\, \mathrm{Tr}[E(\boldsymbol{m})\rho(\boldsymbol{\theta})]$$

An information summary

- What is the goal?

  > Estimator: $\boldsymbol{g}(\boldsymbol{m})$

  > Post-processing error

$$\epsilon(\boldsymbol{m}) = \int d\boldsymbol{\theta}\; p(\boldsymbol{\theta}|\boldsymbol{m})\; \mathcal{D}[\boldsymbol{g}(\boldsymbol{m}), \boldsymbol{\theta}, \mathcal{W}]$$

Relevant in experiments

- How do we get there?

  > Minimise the *overall* uncertainty

$$\bar{\epsilon} = \int d\boldsymbol{\theta} d\boldsymbol{m}\; p(\boldsymbol{\theta}, \boldsymbol{m})\; \mathcal{D}[\boldsymbol{g}(\boldsymbol{m}), \boldsymbol{\theta}, \mathcal{W}]$$

That is, we need to calculate: $\min_{g,E} \bar{\epsilon} = ?$

Relevant for theorists

arXiv:1912.02324

Note that…

- Deviation function for **phases** lying within $\theta_i \in \left[-\frac{L}{2}, \frac{L}{2}\right]$, with $L < 2$
  - Single parameter **square error**:

$$\mathcal{D}[g(\boldsymbol{m}), \theta] \approx [g(\boldsymbol{m}) - \theta]^2$$

  - For multiple parameters, with weighting matrix W:

$$\mathcal{D}[\boldsymbol{g}(\boldsymbol{m}), \boldsymbol{\theta}, \mathcal{W}] \approx \mathrm{Tr}\left\{\mathcal{W}\left[\boldsymbol{g}(\boldsymbol{m}) - \boldsymbol{\theta}\right]\left[\boldsymbol{g}(\boldsymbol{m}) - \boldsymbol{\theta}\right]^{\mathsf{T}}\right\}$$

- A useful approximation (Cramér-Rao asymptotic limit):

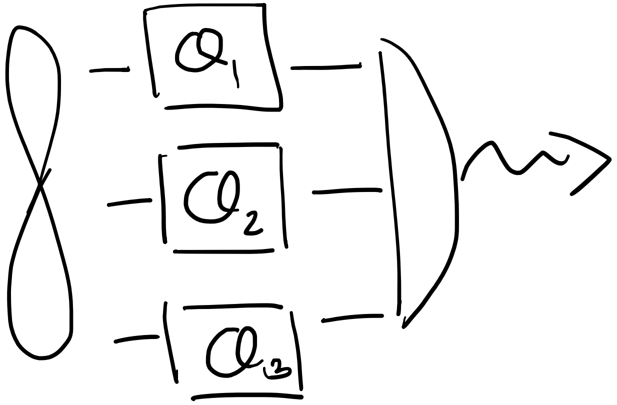
$$\bar{\epsilon}_{\mathrm{mse}} \gtrsim \frac{1}{\mu}\mathrm{Tr}(W F_q^{-1})$$

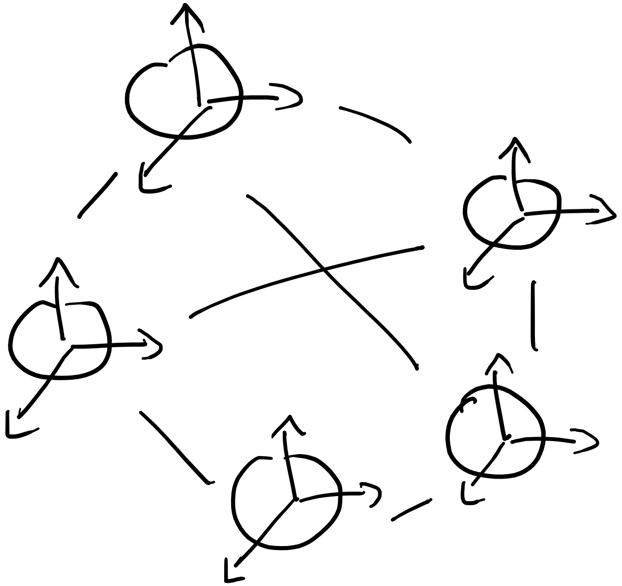
$\mu \gg 1$

arXiv:1912.02324

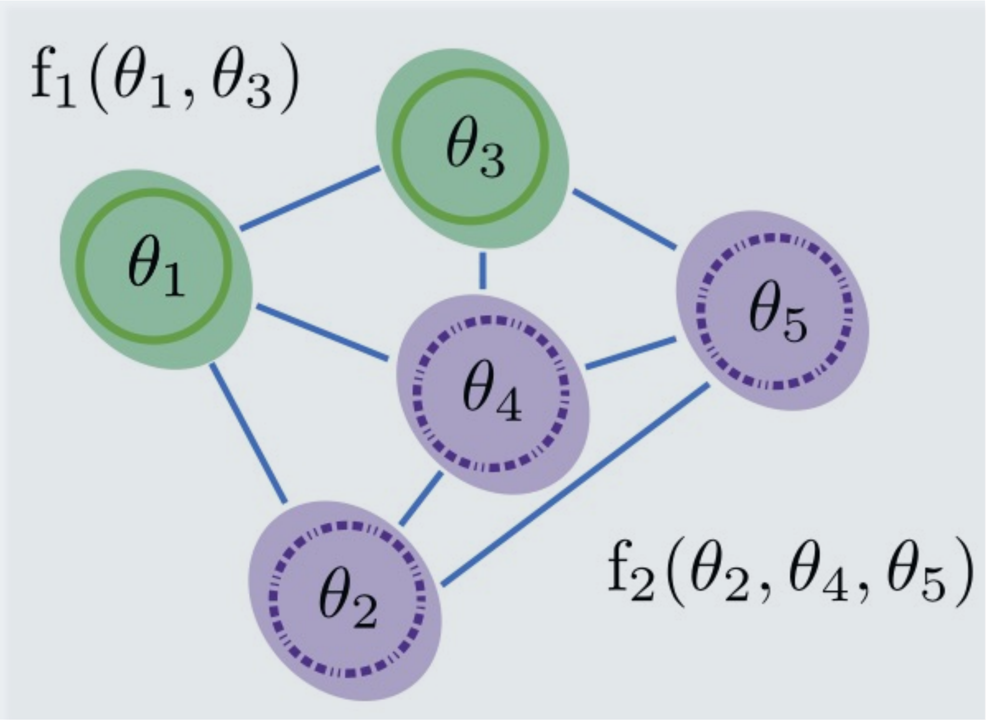
J. Phys. A: Math. Theor. **53** (2020) 344001

Quantum sensing networks: imaging and qubits





Quantum sensing networks: local and global properties



- **Local properties**:
  each individual sensor
- **Global properties**:
  several sensors involved

J. Phys. A: Math. Theor. **53** (2020) 344001

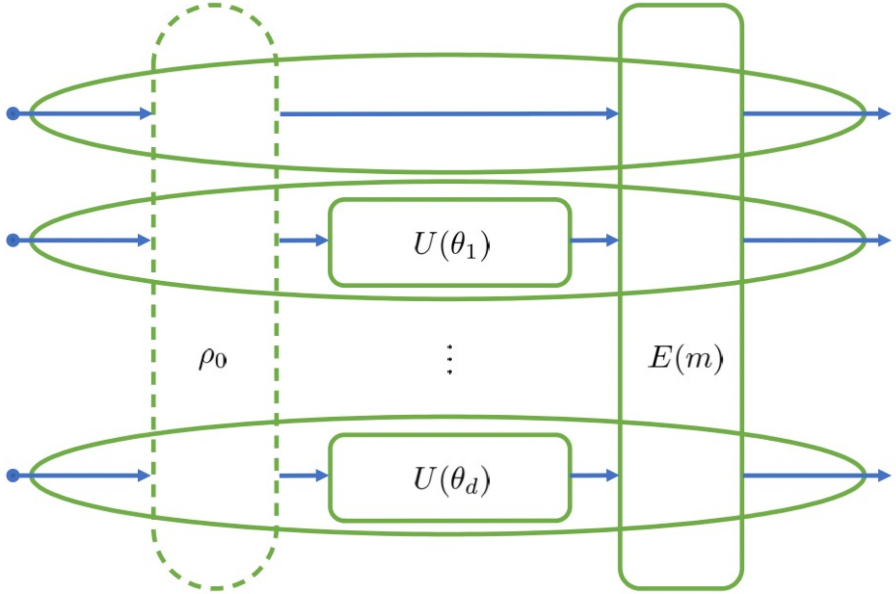
T. J. Proctor, P. A. Knott, and J. A. Dunningham, Networked quantum sensing, arXiv:1702.04271 (2017).

(Discrete) Quantum imaging

- Quantum enhancement by using NOON states

$$|\psi_0\rangle = \frac{1}{\sqrt{d+\alpha^2}}\left(\alpha\,|\bar{n}\,0\cdots0\rangle + \cdots + |0\cdots0\,\bar{n}\rangle\right)$$

P. C. Humphreys, M. Barbieri, A. Datta, and I. A. Walmsley, Quantum enhanced multiple phase estimation, Phys. Rev. Lett. **111**, 070403 (2013).



- Entanglement *not* needed for such an enhancement

$$\rho_0^{\text{ref}} \otimes \rho_0^{(1)} \otimes \cdots \otimes \rho_0^{(d)}$$

$$\rho_0^{(i)} = |\phi_0^{(i)}\rangle\langle\phi_0^{(i)}|$$

$$|\phi_0\rangle = \left[\sqrt{1-\frac{\bar{n}}{N(d+1)}}\,|0\rangle + \sqrt{\frac{\bar{n}}{N(d+1)}}\,|N\rangle\right]$$

P. A. Knott, T. J. Proctor, A. J. Hayes, J. F. Ralph, P. Kok, and J. A. Dunningham, Local versus global strategies in multiparameter estimation, Phys. Rev. A **94**, 062312 (2016).

arXiv:1912.02324

A new multi-parameter quantum bound

$$\bar{\epsilon}_{\text{mse}} \geqslant \sum_{i=1}^{d} w_i \left[\int d\boldsymbol{\theta} p(\boldsymbol{\theta}) \theta_i^2 - \text{Tr}\left(\rho S_i^2\right) \right]$$

❖ For NOON states:

$$\bar{\epsilon}_{\text{mse}} \geqslant \frac{1}{\bar{n}^2} \left[\frac{\pi^2}{3} - \frac{4}{(1+\sqrt{d})^2} \right] \xrightarrow[d \gg 1]{} \frac{1}{\bar{n}^2} \left(\frac{\pi^2}{3} - \frac{4}{d} \right)$$

The local strategy can be as good but not arbitrarily precise!

❖ For the local strategy:

$$\bar{\epsilon}_{\text{mse}} \geqslant [\pi^2/3 - f(N, \bar{n}, d)]/\bar{n}^2$$

(a) if $N = \bar{n}$, then $f(N, \bar{n}, d) = 4d/(1+d)^2$, and

$$\bar{\epsilon}_{\text{mse}} \geqslant \frac{1}{\bar{n}^2} \left[\frac{\pi^2}{3} - \frac{4d}{(1+d)^2} \right] \xrightarrow[d \gg 1]{} \frac{1}{\bar{n}^2} \left(\frac{\pi^2}{3} - \frac{4}{d} \right);$$

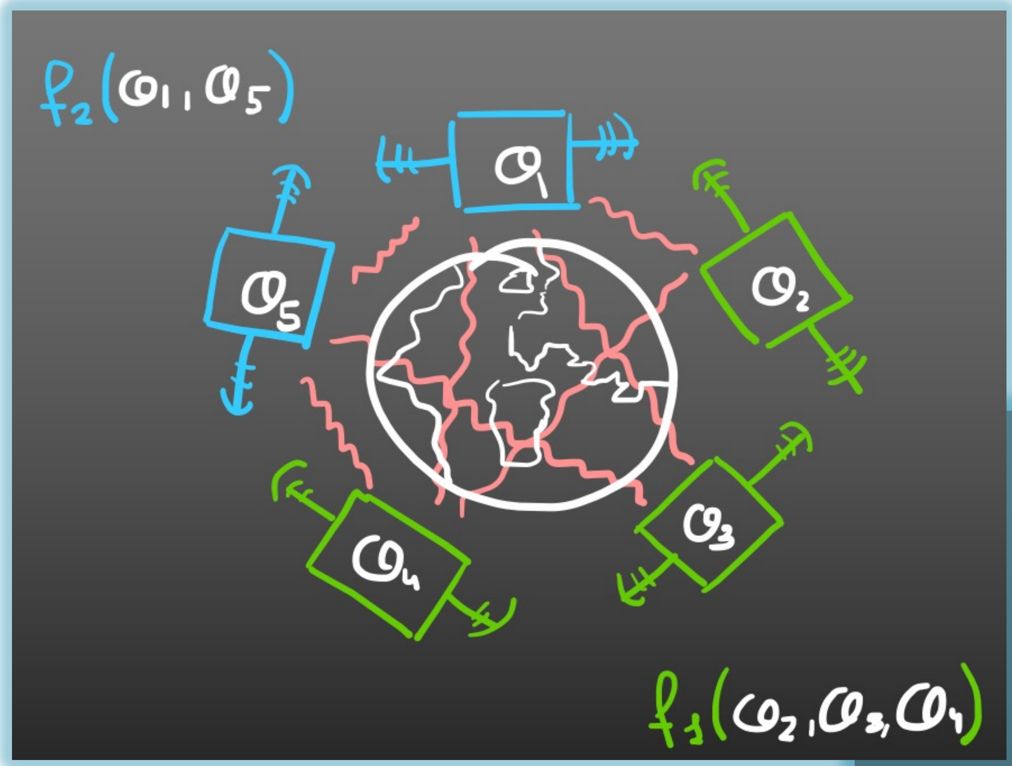
(b) if $N \to \infty$, then $f(N, \bar{n}, d) \to 0$, so

$$\bar{\epsilon}_{\text{mse}} \xrightarrow[N \to \infty]{} \frac{\pi^2}{3\bar{n}^2} = \frac{1}{d} \sum_{i=1}^{d} \Delta\theta_{p,i}^2.$$

The metrological power of vacuum-number superpositions is, at best, equivalent to that of NOON states.

D. Branford and J. Rubio New J. Phys. **23** 123041 (2021)

Qubit sensing network



1) $\rho_0 = |\psi_0\rangle\langle\psi_0|$

$$\downarrow$$

2) $\rho(\vec{\omega}) = e^{-\vec{K}^T \vec{\omega}} \rho_0 \, e^{\vec{K}^T \vec{\omega}}$

$$\left[e^{-\vec{K}^T \vec{\omega}} \equiv e^{-\sigma_z \omega_1/2} \otimes e^{-\sigma_z \omega_2/2} \otimes \dots ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

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* Inter-sensor correlations

$$J_{ij} \equiv \frac{\langle k_i k_j \rangle - \langle k_i \rangle \langle k_j \rangle}{\Delta k_i \, \Delta k_j} \Rightarrow$$

For local strategies, $J_{ij} = 0, \ \forall i,j$

$$\left(\langle * \rangle = \mathrm{Tr}[\rho_0 \, *] ; \ \Delta k_i^2 = \langle k_i^2 \rangle - \langle k_i \rangle^2 \right)$$

sensor-symmetric states

$$\langle k_i k_j \rangle - \langle k_i \rangle \langle k_j \rangle \equiv c$$

$$\langle k_i^2 \rangle - \langle k_i \rangle^2 \equiv v$$

$\forall i,j$ $\Rightarrow$ $J_{ij} \equiv J = \frac{c}{v}$

state $\longleftrightarrow$ (v, J)

($\vec{k}$ fixed)

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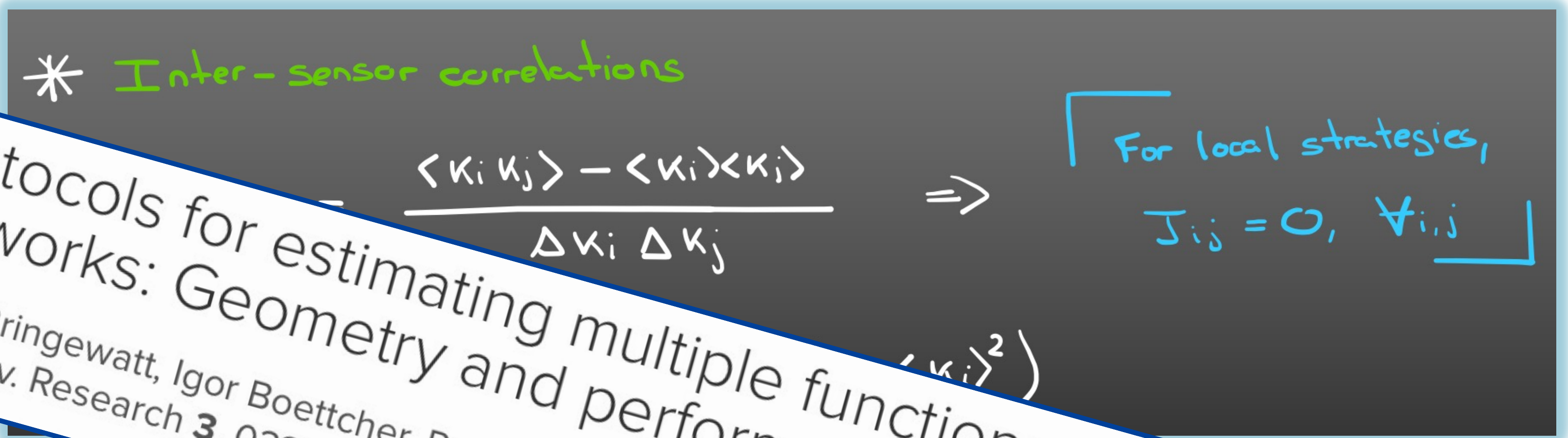
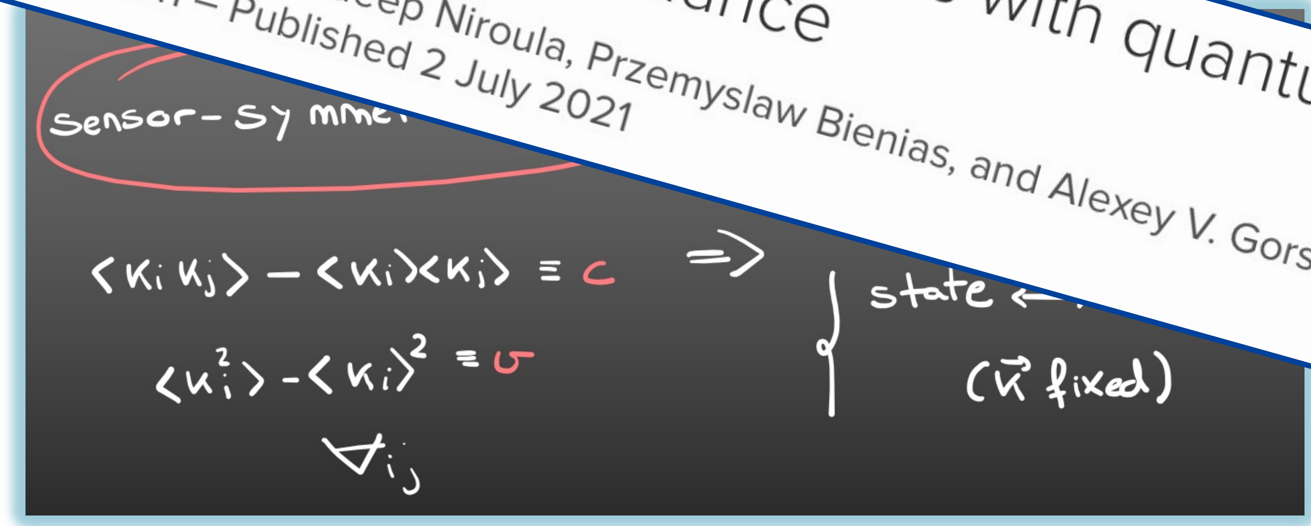




Protocols for estimating multiple functions with quantum sensor networks: Geometry and performance

Jacob Bringewatt, Igor Boettcher, Pradeep Niroula, Przemyslaw Bienias, and Alexey V. Gorshkov
Phys. Rev. Research **3**, 033011 – Published 2 July 2021

J. Phys. A: Math. Theor. **53** (2020) 344001

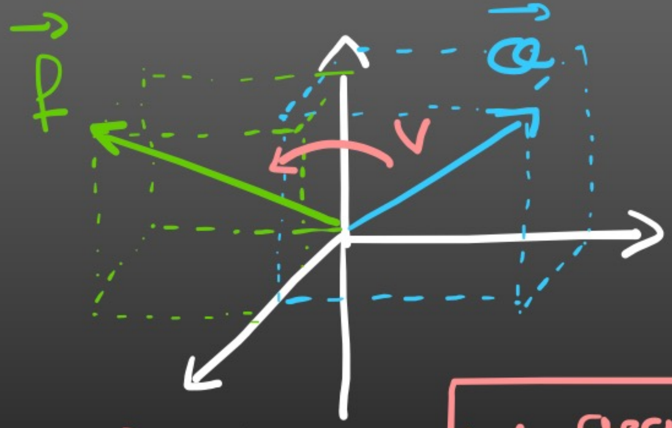
T. J. Proctor, P. A. Knott, and J. A. Dunningham, Networked quantum sensing, arXiv:1702.04271 (2017).

1) Linear functions: exact case

$$\vec{f}(\vec{\theta}) = V^T\vec{\theta} + \vec{a}$$

$$(\;)=(\;\;)(\;)+(\;)$$

$\ell \times 1 \qquad \ell \times d \qquad d \times 1 \qquad \ell \times 1$



$(\vec{a} = 0)$

V ≡ geometric transformation

2) Linear approximation

* general $\vec{f}(\vec{\theta})$
*

⇓

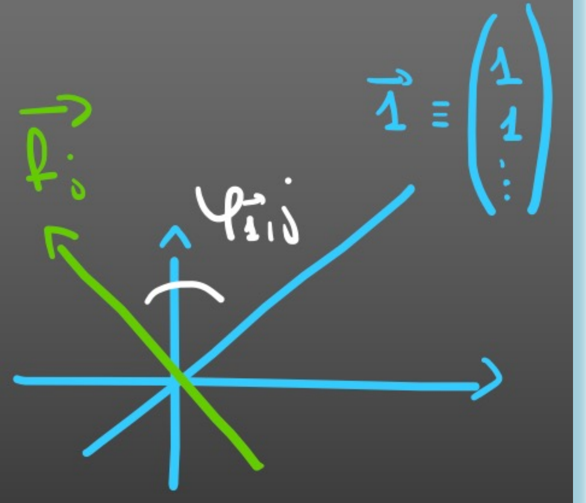
$$\vec{f}(\vec{\theta}) \simeq \vec{f}(\vec{b}) + \sum_{i=1}^{\ell} \frac{\partial \vec{f}(\vec{b})}{\partial \theta_i}(\theta_i - b_i)$$

$$\equiv V^T\vec{\theta} + \vec{a}$$

* Geometric reinterpretation:

$$V = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{f}_1 & \vec{f}_2 & \cdots & \vec{f}_\ell \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \longleftrightarrow f_j(\vec{\omega}) = \sum_{i=1}^{d} V_{ij}\,\omega_i \equiv \vec{f}_j^{\,T} \vec{\omega}$$

$$\Rightarrow \begin{cases} \mathrm{Tr}(WV^TV) = \sum_{j=1}^{\ell} w_j |\vec{f}_j|^2 \\ \mathrm{Tr}(WV^TXV) = \sum_{j=1}^{\ell} w_j |\vec{f}_j|^2 \left[d\cos^2(\varphi_{\vec{1}j}) - 1\right] \end{cases}$$



→ Normalisation term

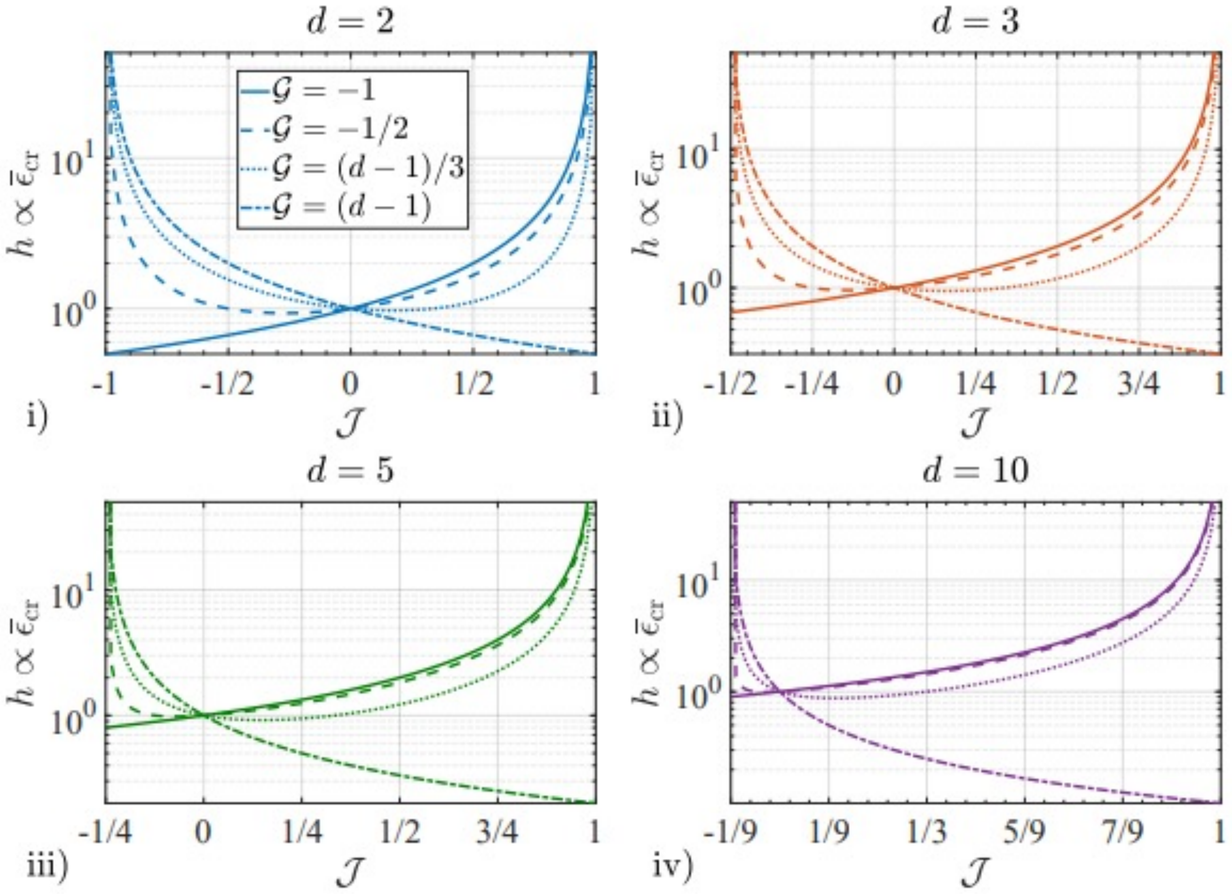
$$\mathcal{N} := \mathrm{Tr}(WV^{T}V)$$

→ Geometry parameter

$$\mathcal{G} := \frac{1}{\mathcal{N}}\mathrm{Tr}[WV^{T}XV]$$

→ Final uncertainty

$$\bar{\epsilon}_{cr} = \frac{\mathcal{N}}{4M\sigma}\underbrace{\frac{[1+(d-\mathcal{G})\mathcal{J}]}{(1-\mathcal{J})[1+(d-1)\mathcal{J}]}}_{=h(\mathcal{J},\mathcal{G},d)}$$



QUESTION: Given (N, G), optimal (υ, J) ?

(N, G) ↓ geometry of linear functions

(υ, J) ↓ state ρ_0 ($\vec{k}$ fixed)

$$\bar{E}_{cr} \geq \frac{N}{M} h(J, G, d)$$

$\upsilon \leq \frac{1}{4}$

⟹ minimisation of h w.r.t. J

$$J_{opt} = \frac{1}{G+2-d}\left[1-\sqrt{\frac{(G+1)(d-1-G)}{d-1}}\right]$$

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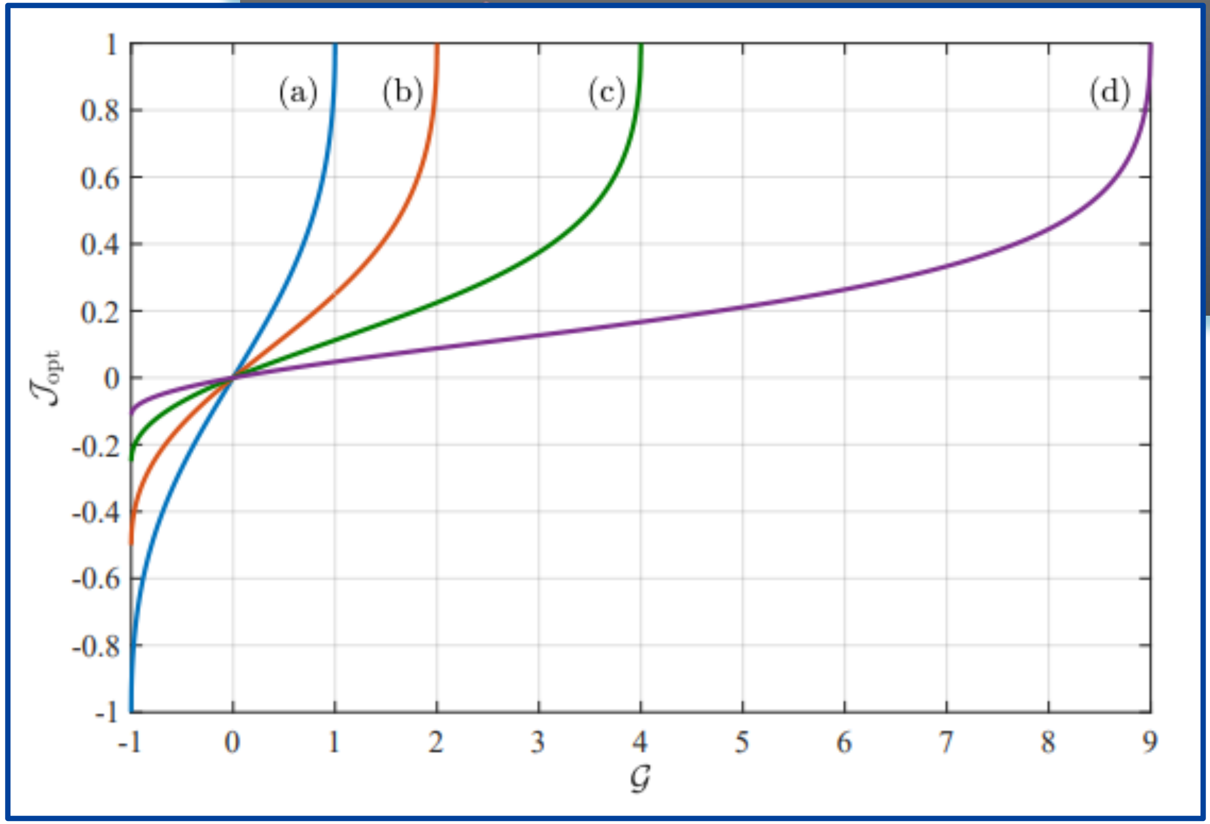
QUESTION: Given $(N, \mathcal{G})$, optimal $(\rho, \mathcal{J})$?

$(N, \mathcal{G})$ → geometry of linear functions

$(\rho, \mathcal{J})$ → state ρ_0 ($\vec{v}$ fixed)

$$\mathcal{J}_{opt} = \frac{1}{\mathcal{G}+2-d}\left[1-\sqrt{\frac{(\mathcal{G}+1)(d-1-\mathcal{G})}{d-1}}\right]$$



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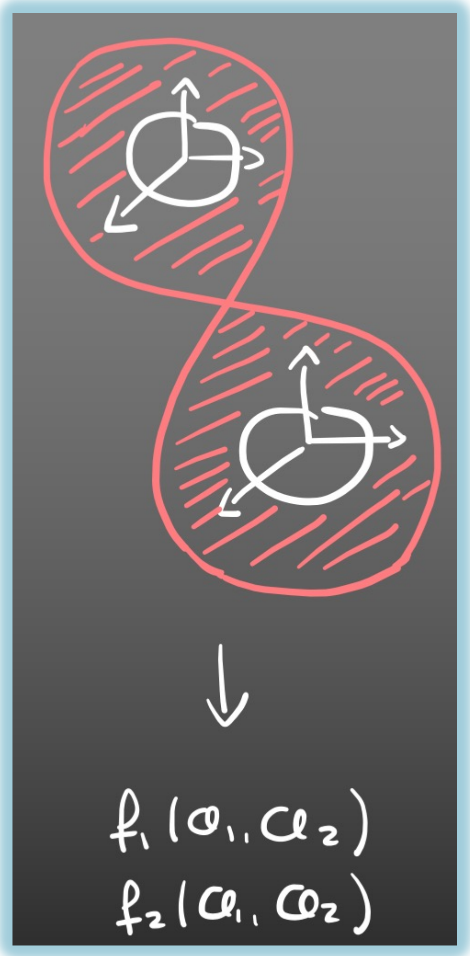
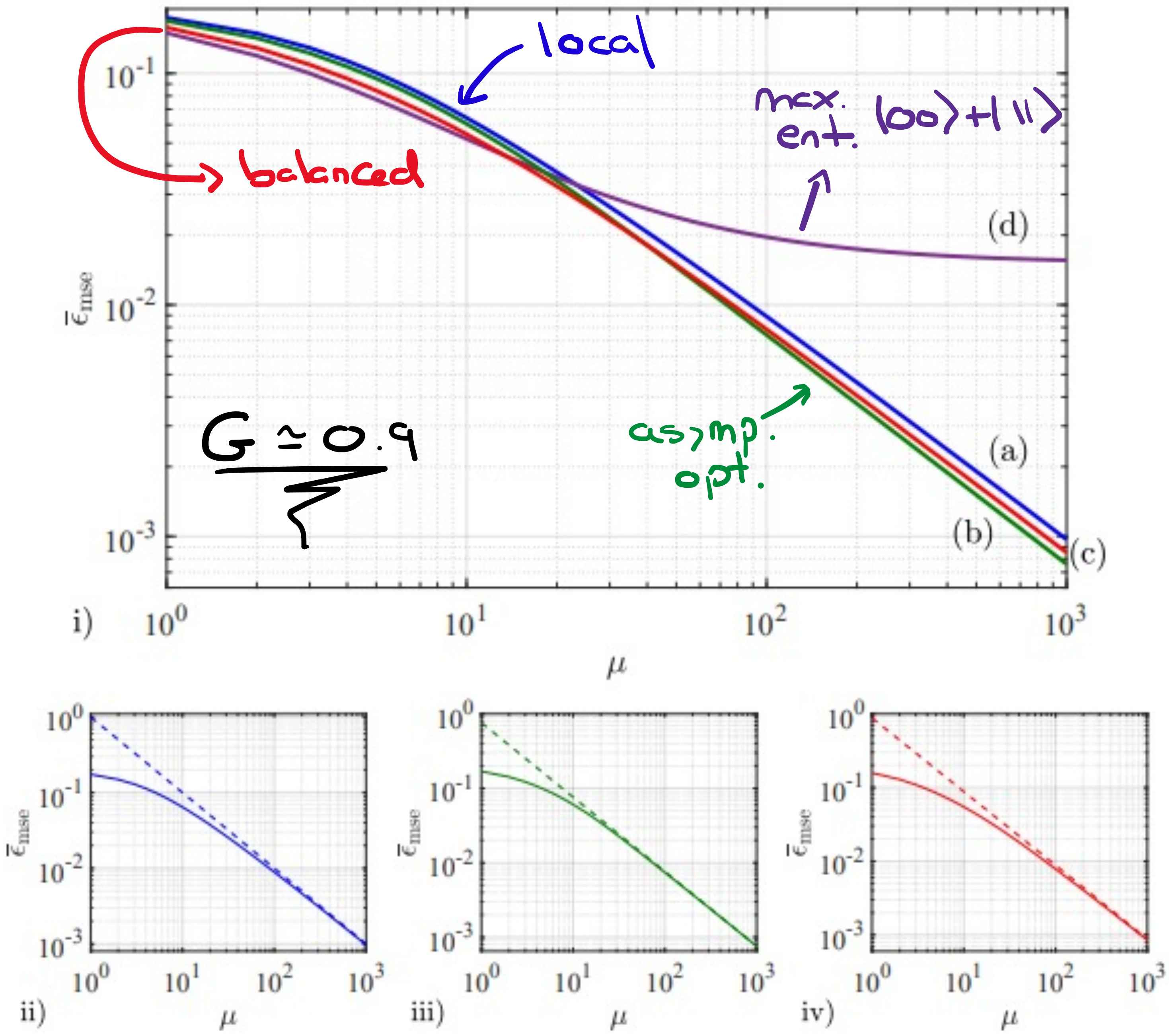


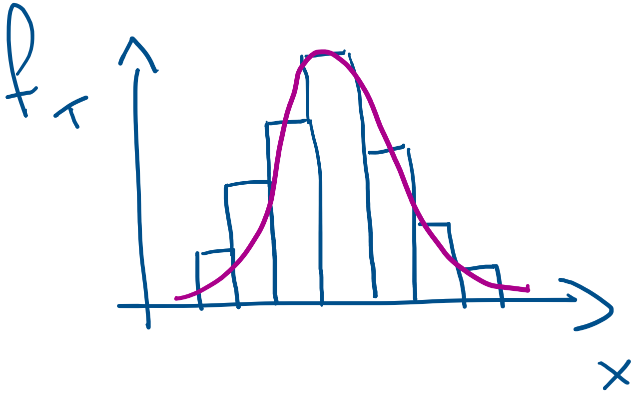
f1(θ1, θ2)
f2(θ1, θ2)



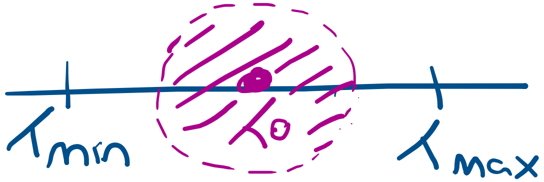
local
balanced
max. ent.
|00⟩+|11⟩
asymp. opt.
G ≃ 0.9
(a)
(b)
(c)
(d)
$\bar{\epsilon}_{\rm mse}$
μ
i)
ii)
iii)
iv)

Quantum Thermometry

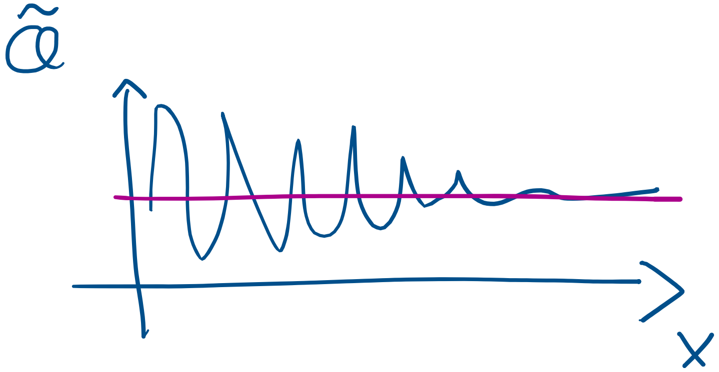
the practical



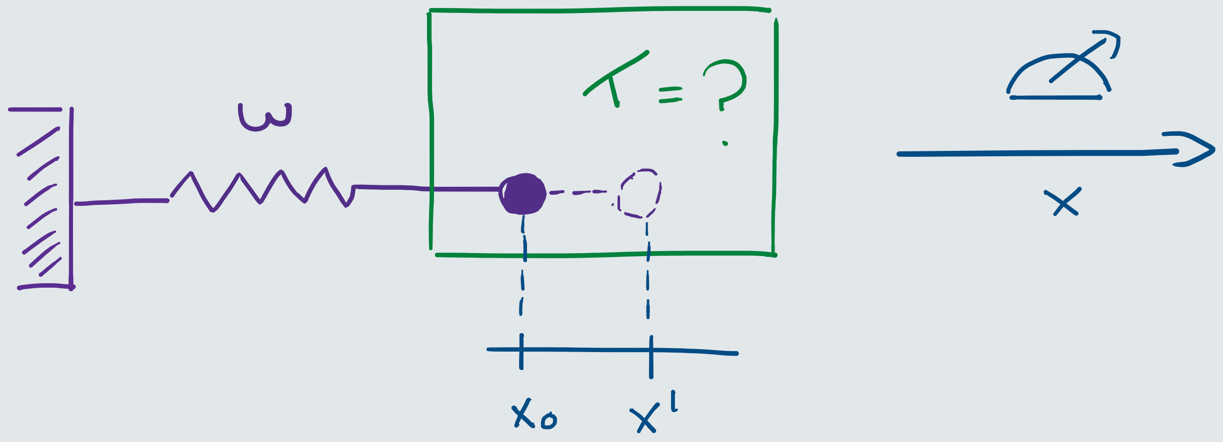
the local



the global



Quantum harmonic oscillator in thermal equilibrium



$$x := q\sqrt{m\omega/\hbar}$$

- θ = hypothesis about the true value of T
- Protocol statistics fully described by:

$$p(x|\theta)dx = \frac{\exp\{-x^2/[2\,\sigma(\theta)^2]\}}{\sqrt{2\pi\sigma(\theta)^2}}\,dx$$

$$\sigma(\theta) = \sqrt{\frac{1}{2}\coth\left(\frac{\hbar\omega}{2k_B\theta}\right)}$$

R. K. Pathria, *Statistical mechanics*

Usual procedure in practice:

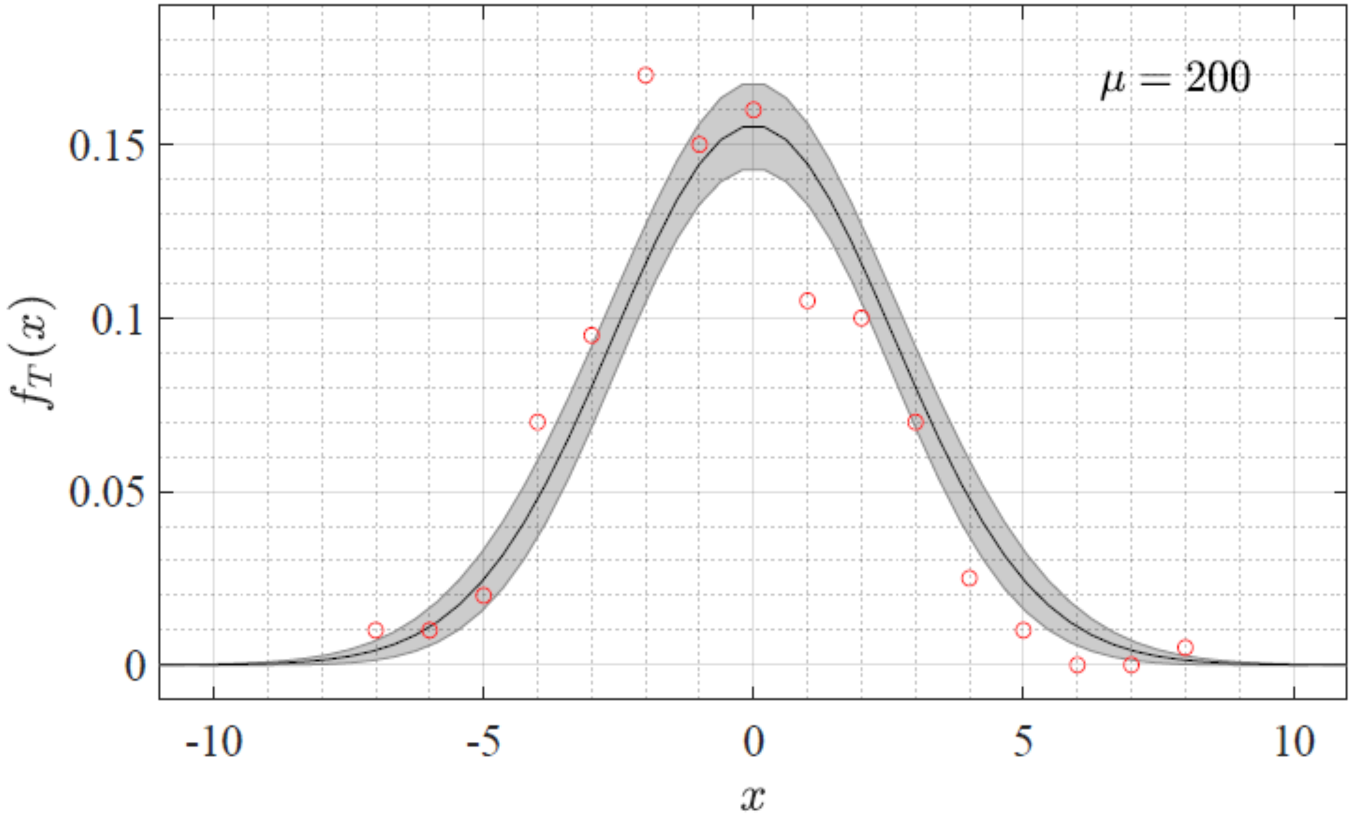
> Measure the position

$\bar{x} = (x_1, x_2, \ldots, x_\mu)$

> Build a position histogram

> Fit the temperature-dependent probability to such histogram

[i.e., to $p(x|T)$]



$\longrightarrow$

$$k_B(\tilde{\theta}_F \pm \Delta\tilde{\theta}_F)/(\hbar\omega) = 7 \pm 1$$

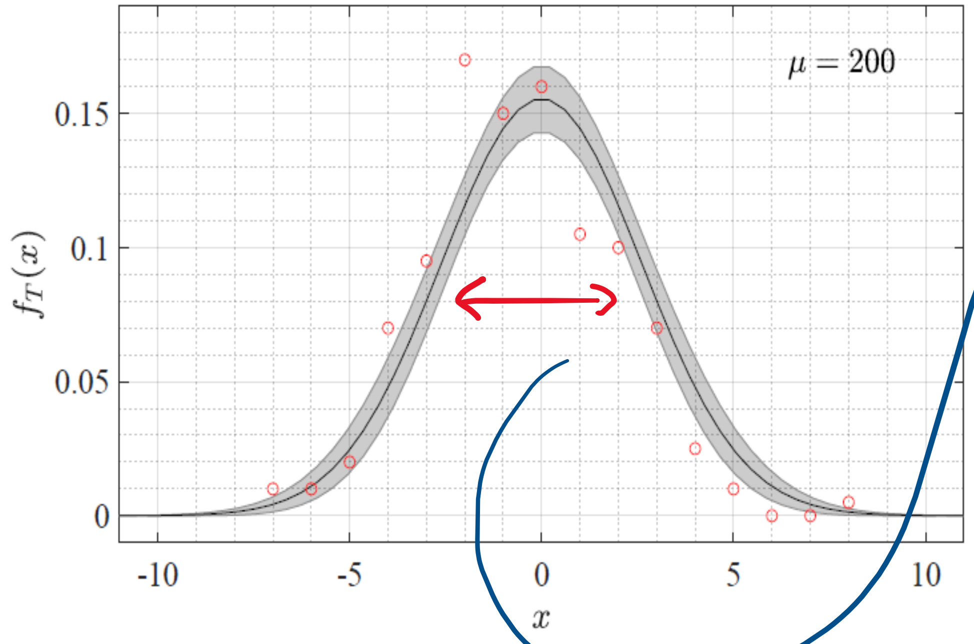
Least squares

Phys. Rev. Lett. **127**, 190402 (2021)

Caveats

- Histogram-based approaches:
  - Bin selection
  - Sufficiently large number of measurements

Local quantum thermometry: how does it help?



- Cramér-Rao bounds:

$$\Delta\tilde{\theta}^2 \geq \frac{1}{\mu\, F(T)} \geq \frac{1}{\mu\, F_q(T)}$$

measurement-dependent Fisher info.

state-dependent Fisher info.

J. Phys. A **52**, 303001 (2019)

Local quantum thermometry: how does it help?

- Direct experimental design:

Given the dynamics (Hamiltonian)

↓

find:
  - state $\rho(T)$
  - measurement M(x)

s.t. $F_q(T)$ is maximum.

- Indirect experimental design:

Given a practical $M(x)$ and a specific state $\rho(T)$

↓

calculate $F(T)$, $F_q(T)$;

  - if $F(T) = F_q(T)$, the scheme is optimal
  - if $F(T) \neq F_q(T)$, keep searching

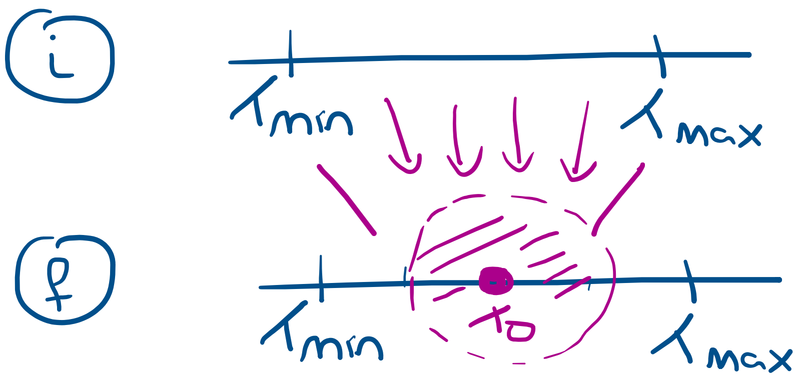
J. Phys. A **52**, 303001 (2019)

Caveats

- Histogram-based approaches:
  - > Bin selection
  - > Sufficiently large number of measurements

- Local quantum thermometry:
  - > Exact but **very** restrictive: exponential family + unbiasedness
  - > Local prior information

  - > Asymptotically large data set



  - > <u>Dependence on true temperature</u>

J. Phys. A **52**, 303001 (2019)

R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kołodyński, Quantum Limits in Optical Interferometry, Progress in Optics **60**, 345 (2015)

A more general starting point: the Bayesian paradigm

- Prior information $\rightarrow \theta \in [\theta_{min}, \theta_{max}]$

    > <u>Maximum ignorance</u> for scale parameters $\rightarrow p(\theta) \propto \frac{1}{\theta}$

- Assessing the (overall) uncertainty of scale parameters: **logarithmic errors**

$$\bar{\epsilon}_{\mathrm{mle}} = \int dE d\theta p(E,\theta) \boxed{\log^2\left[\frac{\tilde{\theta}(E)}{\theta}\right]}$$

generalised **relative error** or
noise-to-signal ratio

$$\min_{\tilde{\theta}(E)} \bar{\epsilon}_{mle} = ?$$

Phys. Rev. Lett. **127**, 190402 (2021)

A two-line solution

- **Optimal rule to post-process measurement outcomes into a temperature reading**
  - Universally valid

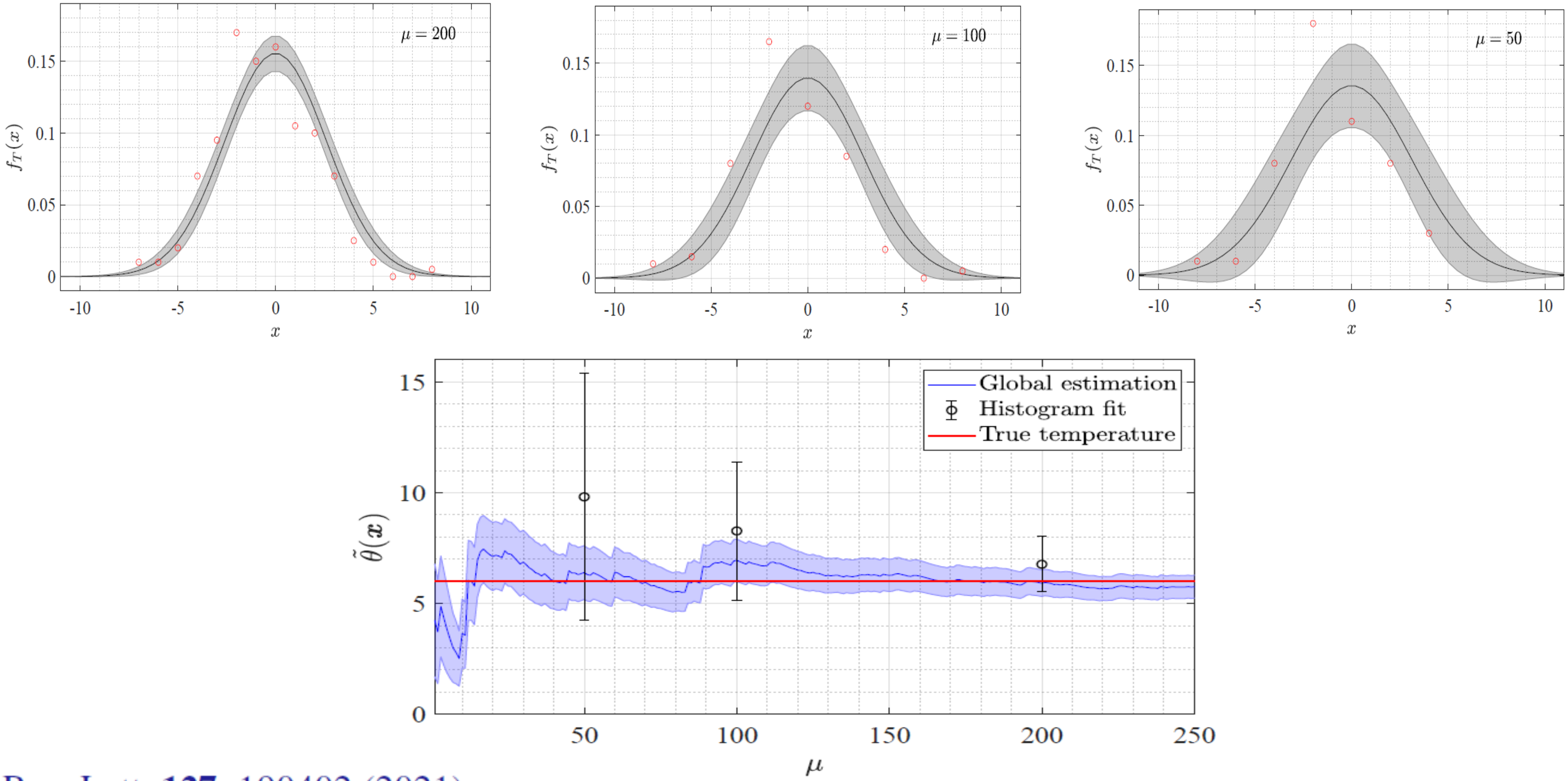
$$\frac{k_B \tilde{\vartheta}(E)}{\varepsilon_0} = \exp\left[\int d\theta\, p(\theta|E) \log\left(\frac{k_B \theta}{\varepsilon_0}\right)\right]$$

- **Minimum** uncertainty *overall* (not just a bound)
  - Useful <u>to study fundamental limits</u> to the precision
  - University valid *for a given measurement*
  - Not just a bound!

$$\bar{\epsilon}_{\text{mle}} \gtrsim \bar{\epsilon}_{\text{opt}} = \bar{\epsilon}_p - \mathcal{K}$$

Phys. Rev. Lett. **127**, 190402 (2021)

Revisiting the harmonic oscillator in thermal equilibrium



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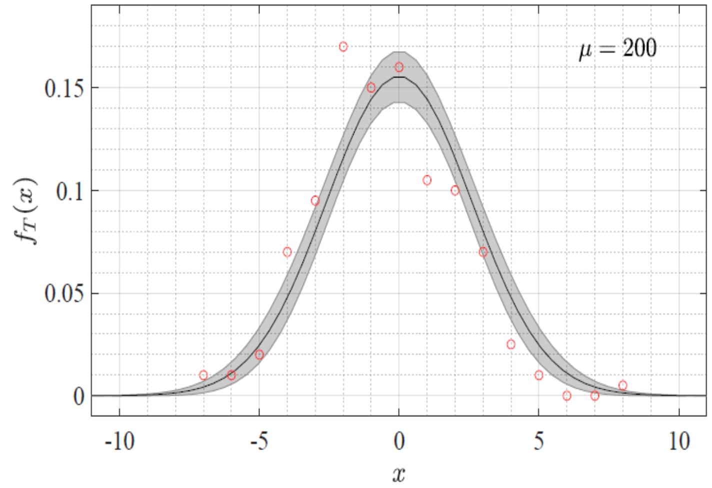
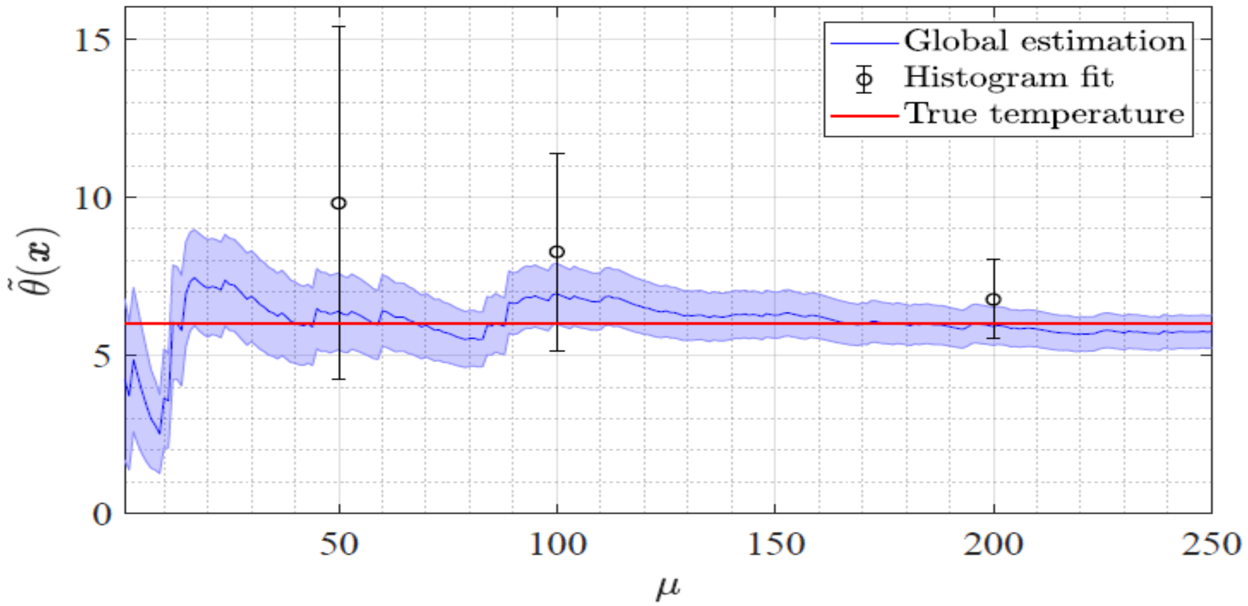
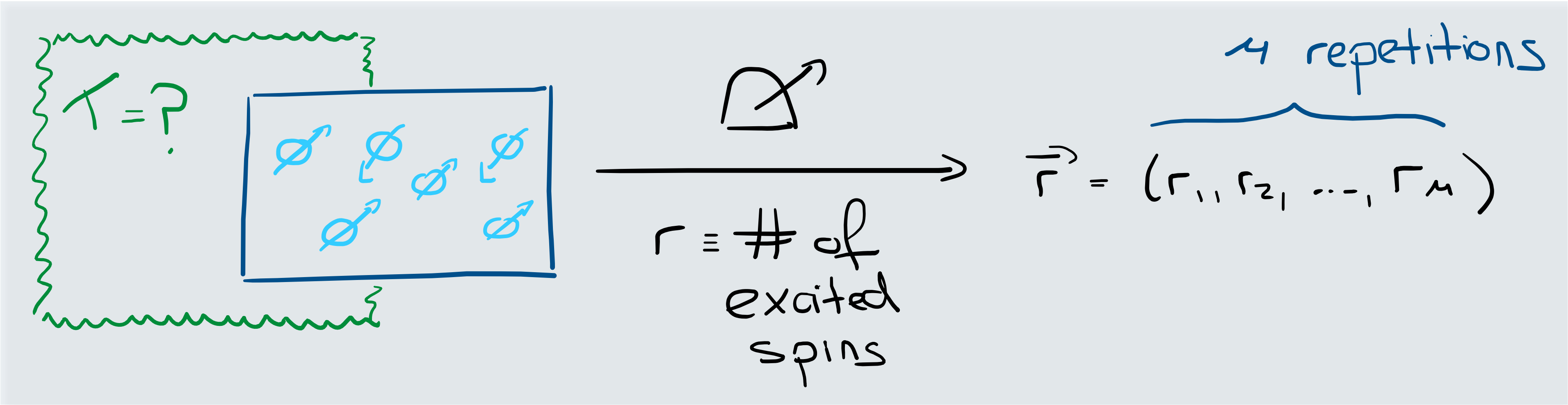


- Least square method: biased for finite statistics
- <u>Bayesian approach</u>: **as good or better than traditional methods**



Phys. Rev. Lett. **127**, 190402 (2021)

The perils of local thermometry: non-interacting spin-1/2 gas

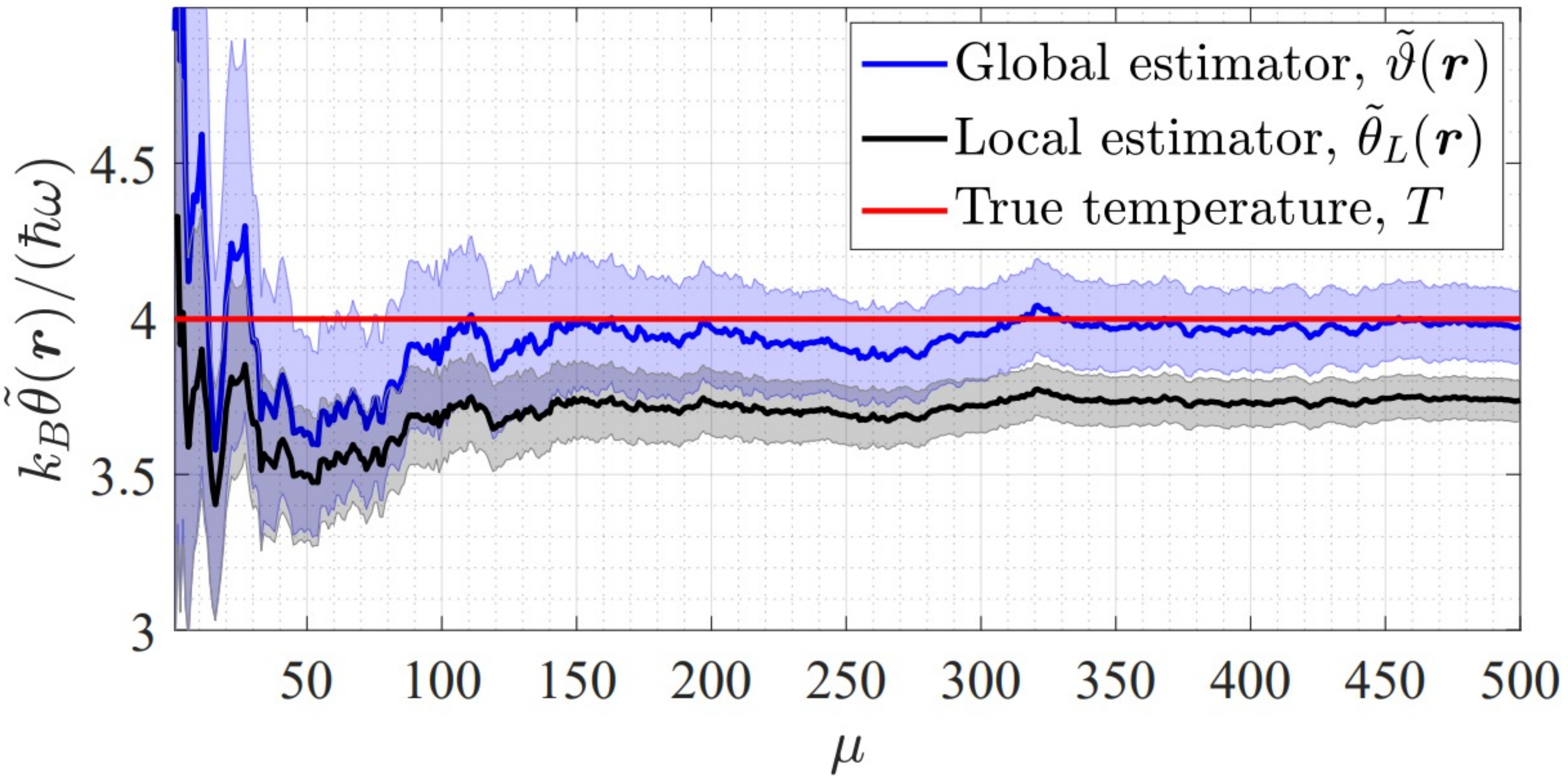


- Prior information: $p(\theta) \propto \frac{1}{\theta}$; $\frac{k_B T}{\hbar\omega} \in [0.1, 10]$

- Measurement information: $p(r|\theta) = \binom{n}{r} \frac{\exp[-r\hbar\omega/(k_B\theta)]}{Z[\hbar\omega/(k_B\theta)]}$

Phys. Rev. Lett. **127**, 190402 (2021)

The perils of local thermometry: non-interacting spin-1/2 gas



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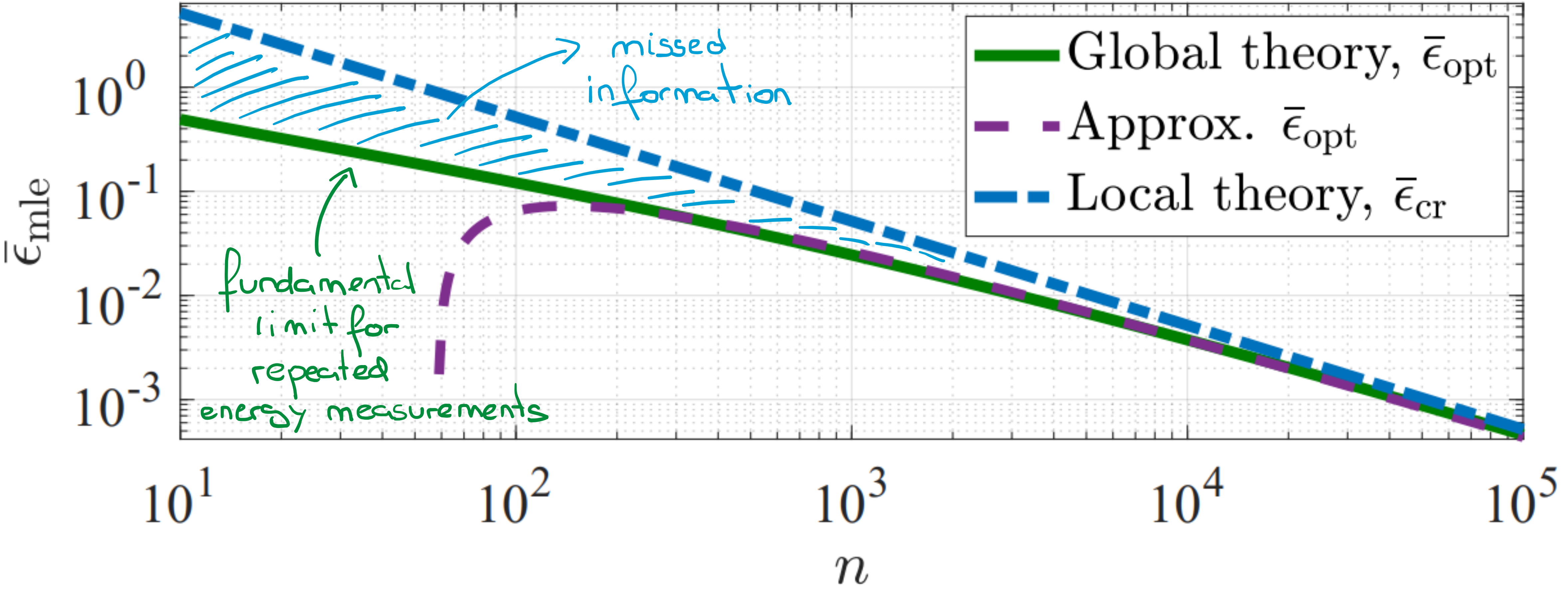
The perils of local thermometry: non-interacting spin-1/2 gas



Phys. Rev. Lett. **127**, 190402 (2021)

Quantum metrology of scale parameters

What is a scale parameter?

Examples:

- temperature: $\frac{E}{k_B T}$
- (Inverse of) Poisson rate: $kt = \frac{t}{1/k}$
- (Inverse of) decay rates: $\gamma t = \frac{t}{1/\gamma}$

Definition:

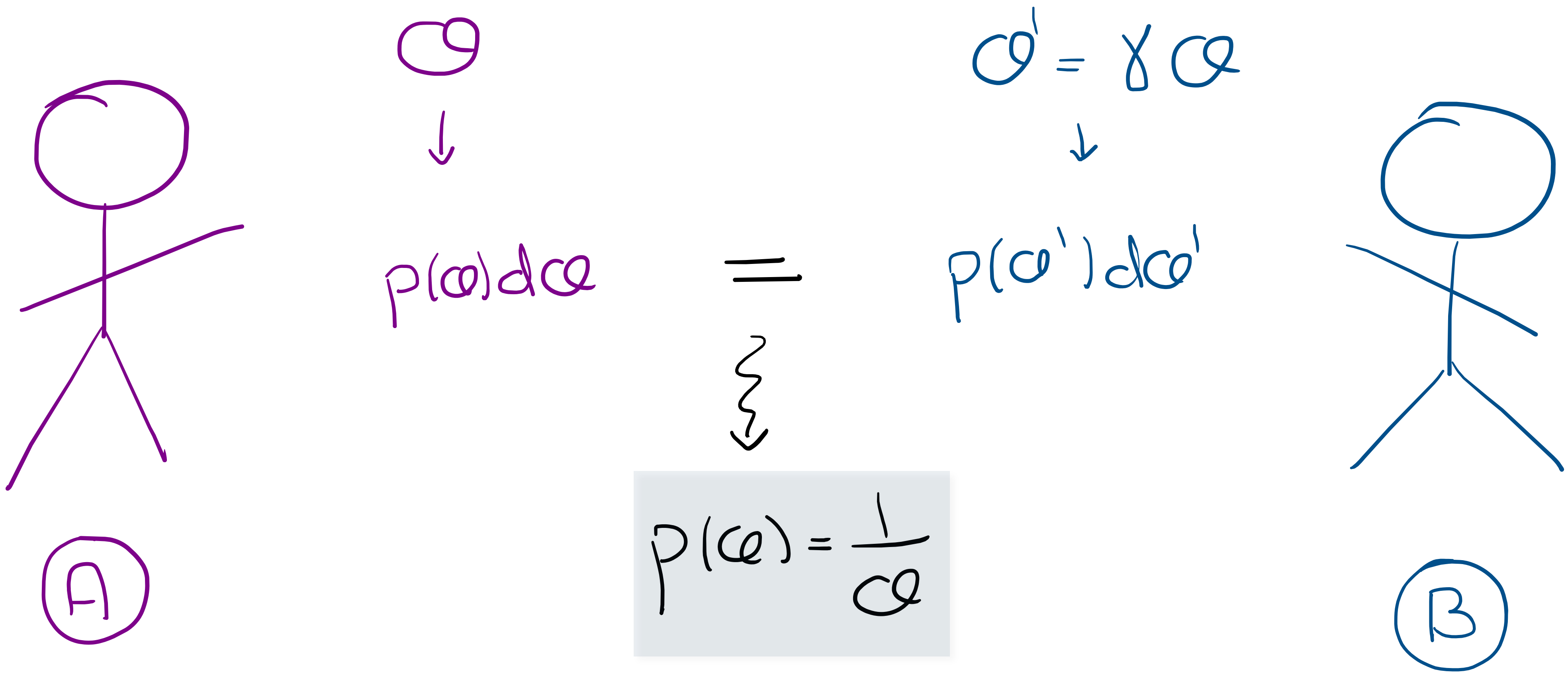
$$\left.\begin{array}{l} Y \text{ is 'large' when } Y/\Theta \gg 1 \\ Y \text{ is 'small' when } Y/\Theta \ll 1 \end{array}\right\} \Rightarrow \Theta \equiv \text{scale parameter}$$

The **key symmetry**: scale invariance

$$\begin{array}{l} Y \mapsto Y' = \gamma Y \\ \Theta \mapsto \Theta' = \gamma \Theta \end{array} \rightarrow \quad Y'/\Theta' = Y/\Theta$$

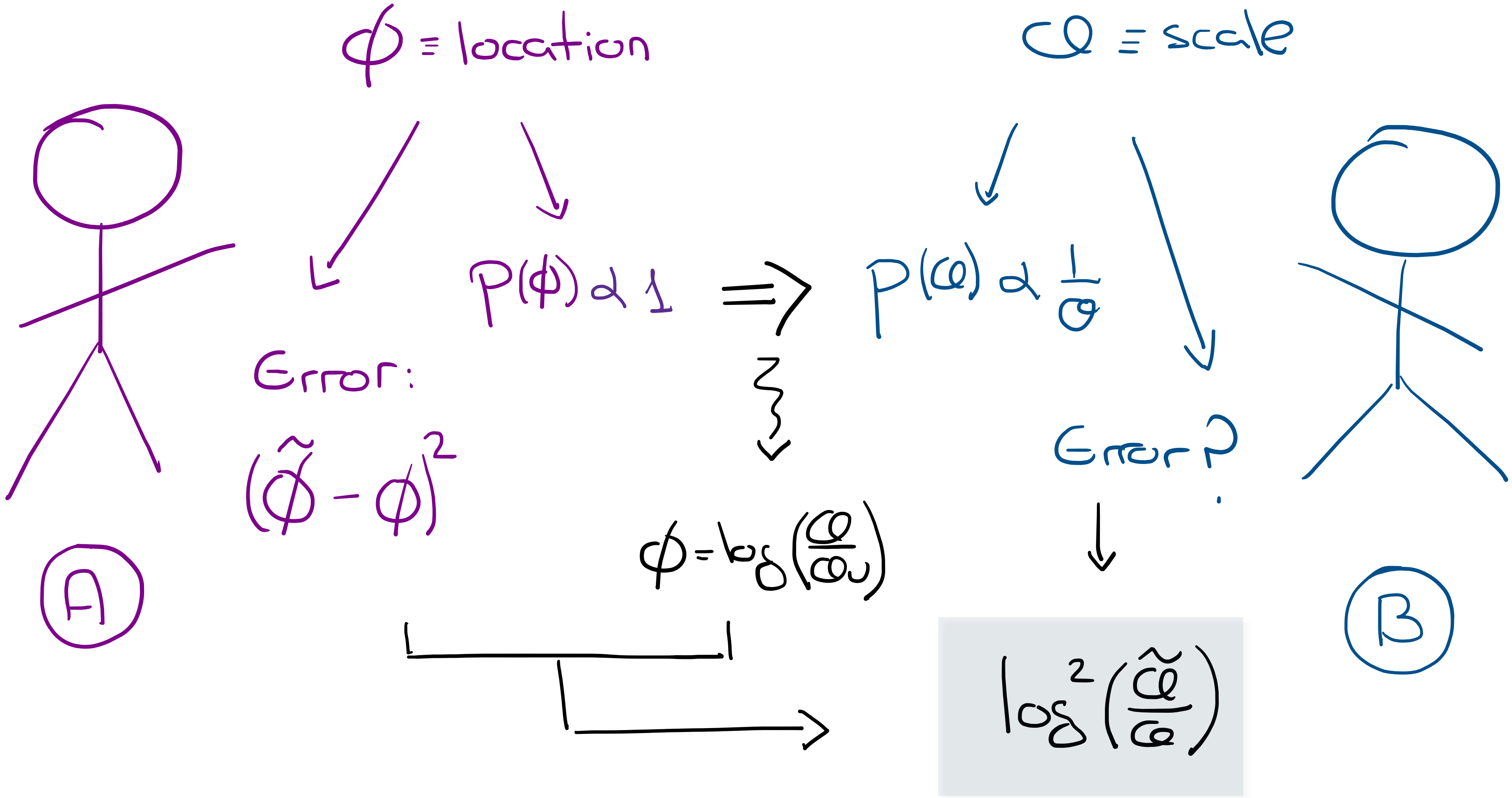
arXiv:2111.11921

Maximum ignorance about scale parameters



E. T. Jaynes, Prior probabilities, IEEE Transactions on Systems and Cybernetics **4**, 227 (1968)

Why logarithmic errors?



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Quantum scale estimation: statement of the problem

Using the Born rule,

$$\bar{\epsilon}_{\mathrm{mle}} = \mathrm{Tr}\left\{ \int dx\, M(x)\, W[\tilde{\theta}(x)] \right\}; \quad W[\tilde{\theta}(x)] = \int d\theta\, p(\theta)\, \rho(\theta) \log^2\left[\frac{\tilde{\theta}(x)}{\theta}\right]$$

measurement ($M(x)$); given! ($p(\theta)\,\rho(\theta)$); prior ($p(\theta)$); state ($\rho(\theta)$); estimator ($\tilde{\theta}(x)$)

Our goal is to find the minimum:

$$\min_{\tilde{\theta}(x),\, M(x)} \mathrm{Tr}\left\{ \int dx\, M(x)\, W[\tilde{\theta}(x)] \right\} = \bar{\epsilon}_{\min}$$

arXiv:2111.11921

Optimal quantum strategy

$$\mathcal{S} = \int ds\, |s\rangle\langle s|\, s$$

Calculate as ⟶

$$\mathcal{S}\varrho_0 + \varrho_0\mathcal{S} = 2\varrho_1$$

$$\varrho_k = \int d\theta\, p(\theta)\rho(\theta)\log^k\left(\frac{\theta}{\theta_u}\right)$$

<u>Optimal measurement</u>

$$\mathcal{M}(s) = |s\rangle\langle s|$$

<u>Optimal estimator</u>

$$\tilde{\vartheta}(s) = \theta_u \exp\left[\int d\theta\, p(\theta|s)\log\left(\frac{\theta}{\theta_u}\right)\right]$$

<u>Experimental error</u>

$$\epsilon(s) = \int d\theta\, p(\theta|s)\, \log^2\left[\frac{\tilde{\vartheta}(s)}{\theta_u}\right]$$

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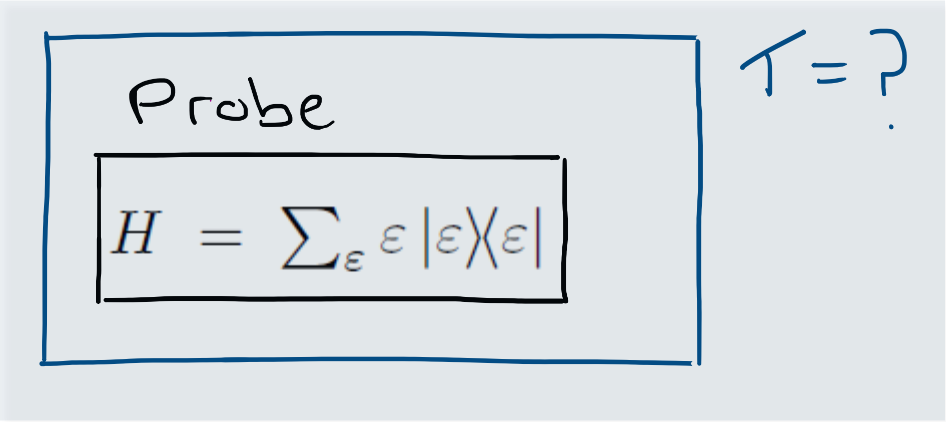
Revisiting equilibrium quantum thermometry



$$\equiv \rho(\theta) = \frac{\exp[-H/(k_B\theta)]}{\mathrm{Tr}\{\exp[-H/(k_B\theta)]\}}$$

$$\mathcal{S} = \sum_\varepsilon |\varepsilon\rangle\langle\varepsilon| \frac{R_{\varepsilon,1}}{R_{\varepsilon,0}}$$

- For thermal states, **energy measurements are universally optimal**
- The optimal measurement may sometimes be implemented in the laboratory

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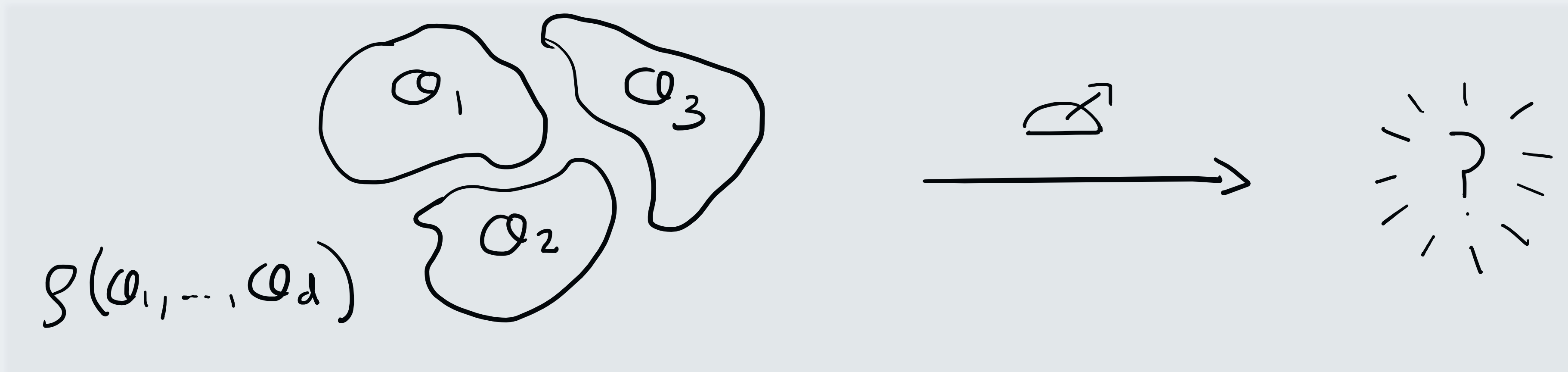
Towards scale-invariant multi-parameter schemes



$$\bar{\epsilon}_{\text{mle}} \geq \frac{1}{d}\sum_{i=1}^{d}\left[\int d\boldsymbol{\theta}\, p(\boldsymbol{\theta}) \log^2\left(\frac{\theta_i}{\theta_{u,i}}\right) - \text{Tr}(\rho_{0,i}\mathcal{S}_i^2)\right]$$

- Not saturable when $[S_i, S_j] \neq 0$
- Quantum compatibility: prior- and uncertainty-dependent

arXiv:2111.11921

A. Luis, Complementarity for Generalized Observables, Phys. Rev. Lett. **88**, 230401 (2002).

Phases, locations and scales

Type of parameter	phase	location	scale
General support	$0 \leq \theta < 2\pi$	$-\infty < \theta < \infty$	$0 < \theta < \infty$
Symmetry	$\theta \mapsto \theta' = \theta + 2\gamma\pi,\ \gamma \in \mathbb{Z}$	$\theta \mapsto \theta' = \theta + \gamma,\ \gamma \in \mathbb{R}$	$\theta \mapsto \theta' = \gamma\theta,\ \gamma \in \mathbb{R}_*^+$
Maximum ignorance	$p(\theta) = 1/2\pi$	$p(\theta) \propto 1$	$p(\theta) \propto 1/\theta$
Deviation function $\mathcal{D}[\tilde{\theta}(x), \theta]$	$4\sin^2\{[\tilde{\theta}(x) - \theta]/2\}$	$[\tilde{\theta}(x) - \theta]^2$	$\log^2[\tilde{\theta}(x)/\theta]$

An attractive perspective:

- > **Elementary quantities** (each its own quantum estimation theory)
- > Multi-parameter estimation with mixed models?

What have we learnt?

- There is, in networked quantum sensing, …
  - > … a **fundamental link between correlations and geometry**
  - > … a trade-off between the asymptotic and non-asymptotic precisions
  - > ... a rich and unexplored area within limited-data metrology, which requires Bayesian techniques by construction

- Quantum thermometry *à la Bayes…*
  - > … is very general (minimal assumptions)
  - > … is **experimentally friendly**, as it provides
    - > a universal map from data sets to optimal estimates
    - > a clear and direct assessment of experimental errors
  - > … is reliable (simulations) and **works in experiments**
  - > … provides the key mathematics for the metrology of scale parameters

- Quantum scale estimation …
  - > … establishes a framework for the most precise estimation that the laws of quantum mechanics allow for scale parameters
  - > … closes an important gap in quantum metrology
  - > … provides a fundamental picture: **phases, locations and scales**

Thank you for your attention

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J. Phys. A: Math. Theor., 53 344001 (2020)
JPhys. Rev. A 101, 032114 (2020)

If you have any question or comment:

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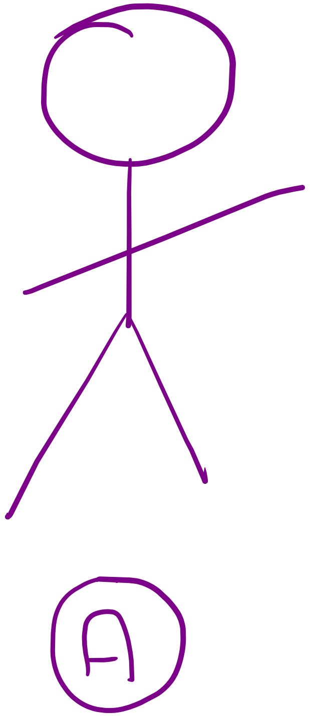
Supplementary material

Why logarithmic errors?

Prior range: $\omega \in [0.01, 100]$; scale



$p(\omega) \propto 1$

$\downarrow$

$\int d\omega\, p(\omega) (\tilde{\omega} - \omega)^2$

$\downarrow$

$\tilde{\omega} = \int d\omega\, p(\omega)\, \omega$

~~$\simeq 50$~~

$p(\omega) \propto \frac{1}{\omega}$

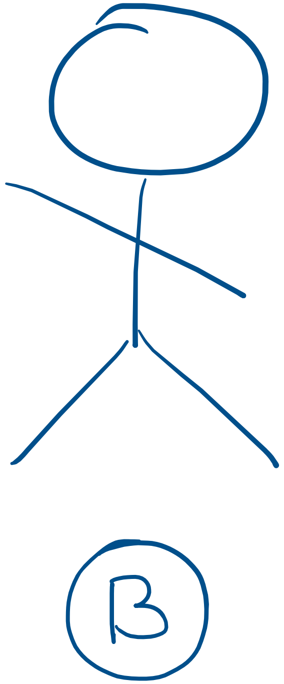
$\downarrow$

$\int d\omega\, p(\omega) \log^2\left(\frac{\tilde{\omega}}{\omega}\right)$

$\downarrow$

$\tilde{\omega} = \exp\left[\int d\omega\, p(\omega) \log(\omega)\right]$

$= 1$ ✓



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Some theoretical consequences: quantum observables

$$\hat{\Theta} = \theta_u \exp(\mathcal{S}) = \ldots = \int ds\, \mathcal{M}(s)\, \tilde{\vartheta}(s)$$

quantum measurement

scale values

D1. The initial state and the associated parameter encoding, both captured by $\rho(\theta)$.

D2. The prior information, represented by $p(\theta)$.

D3. The fact that scale uncertainties are quantified using the mean logarithmic error $\bar{\epsilon}_{\text{mle}}$.

arXiv:2111.11921

Some theoretical consequences: quantum observables

- S. Personick, Application of quantum estimation theory to analog communication over quantum channels, IEEE Transactions on Information Theory **17**, 240 (1971)

  → phase/location observable

- C. W. Helstrom, *Quantum Detection and Estimation Theory*
  A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*

  → phase and time observables

  → position and momentum observables (estimation-theoretic)

- arXiv:2111.11921
  Phys. Rev. Lett. **127**, 190402 (2021)

  → scale observable (e.g., temperature)